Ion Heating with Beating Electrostatic Waves

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The nonlinear interaction of a magnetized ion with two beating electrostatic waves (BEW) whose frequencies differ by a cyclotron harmonic can lead, under some conditions [Phys. Rev. E **69**, 046402 (2004)], to vigorous acceleration for an ion with arbitrarily low initial velocity. When applied to an ensemble of ions, this mechanism promises enhanced heating over single electrostatic wave (SEW) heating for comparable wave energy densities. The extension of single ion acceleration to heating (SEWH and BEWH) of an ensemble of initially thermalized ions was carried out to compare the processes. Using a numerical solution of the Vlasov equation as a guideline, an analytical expression for the heating level was derived with Lie transforms and was used to show BEWH's superiority over all parameter space.

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In 1998, Benisti *et al.* [1] proposed that when two electrostatic waves, propagating perpendicularly to a uniform magnetic field satisfy the beating criterion, $\omega_2 - \omega_1 = n\omega_c$, where ω_1, ω_2 are the wave frequencies, n is an integer number, and ω_c is the ion cyclotron frequency, some magnetized ions with initial velocities arbitrarily lower than the phase velocities of the exciting waves are subject to vigorous acceleration. Spektor and Choueiri [2], while deriving the necessary and sufficient conditions for this acceleration, subsequently showed analytically that there is no velocity threshold for targeting ions with this nonlinear mechanism. It is for this reason that the beating electrostatic waves heating (BEWH) of plasma is expected to be more efficient than resonant, single electrostatic wave heating (SEWH) for comparable wave energy densities and why it represents an attractive alternative for plasma heating in applications such as fusion devices and spacecraft plasma propulsion. So far, published analytical studies [1–3] of BEW have been restricted to the case of a single ion interacting with a pair of electrostatic waves. Extending this analysis from acceleration of a single ion to the heating of an ensemble of ions is of both fundamental and practical importance to plasma heating. In this Letter we carry out this extension in an effort to answer the following questions: Does BEWH outperform SEWH for equal wave energy densities? Is it always the superior process? And if yes, given experimental constraints, can we predict the performance of BEWH?

We first establish the dependence of ion heating on wave number and frequency by numerically solving the Vlasov equation for SEWH and BEWH. Using this as a guideline, we derive an analytical expression with Poincaré cross sections and Lie transform theory that allows a comparison between SEWH and BEWH for arbitrary wave parameters.

We follow Gibelli *et al.* [4] in assuming the ions are collisionless and the waves are uninfluenced by the ion dynamics. The first assumption restricts our analysis to plasmas where heating time scales are faster than collisions. The second limits our scope to small perturbations as it precludes self-consistent effects that would limit the maximum amplitude of the waves [5]. In spite of these restrictions, assuming the waves are uninfluenced by the particle dynamics still allows an approximation for the average result of the stochastic energization of an ensemble [4]. Moreover, this assumption enables us to examine BEWH and SEWH in a general sense: while a self-consistent simulation requires we specify plasma characteristics such as the dielectric response, assuming the waves are uninfluenced by ion dynamics results in a Hamiltonian formulation where the wave parameters are independent variables. We thus can compare SEWH and BEWH over all wave parameter space without specifying a plasma mode. With this in mind, we proceed by formulating the problem in terms of the normalized Vlasov equation with electrostatic waves propagating perpendicularly to a uniform magnetic field:

$$0 = \frac{\partial f}{\partial \tau} + V_X\left[\frac{\partial f}{\partial X} - \frac{\partial f}{\partial V_Y}\right] + \frac{\partial f}{\partial V_X}\left[V_Y + \sum \varepsilon_i \sin(\kappa_i X - \nu_i \tau)\right], \qquad (1)$$

where $i = 1, 2$ and we have normalized the physical variables such that $X = x/\bar{r}_L$, $V_{X,Y} = v_{x,y}/\bar{v}_\perp$, $\nu_i = \omega_i/\omega_c$, $\tau = \omega_c t$, $\varepsilon_i = (qE_i)/(m\bar{r}_L\omega_c^2)$, $\kappa_i = k_i\bar{r}_L$, and $\tau = \omega_c t$. Here E_i is the electric field amplitude of the ith wave; q and m are the charge and mass of the ion, respectively, $\bar{r}_L$ is the root mean squared value of the Larmor radius in the initial ensemble, and $\bar{v}_\perp$ is the rms value of the initial perpendicular velocity. In the summation above, we define $\nu_1 = \nu$ for SEWH and $\nu_1 = \nu$, $\nu_2 = \nu + 1$ for BEWH as this was found [1] to yield the greatest single ion acceleration. In order to ensure equal energy densities for each case, we define $\varepsilon_1 = \epsilon_0$ and $\varepsilon_2 = 0$ for SEWH and $\varepsilon_1 = \varepsilon_2 = \epsilon_0/\sqrt{2}$ for BEWH. Finally, we note that for two waves, $\kappa_1 \neq \kappa_2$ has been shown to have an impact on the level of single ion energization [1]. However, it is also evident from previous studies that if $\|\kappa_2 - \kappa_1\|/\kappa_1 \ll 1$—a restriction valid for large group velocities—the resulting acceleration is on par with $\kappa_1 = \kappa_2$. In order to simplify our analysis then and invoke the results of previous BEW single ion work [2], we define $\kappa_1 = \kappa_2 = \kappa$.

With these constraints, we first solve Eq. (1) using a Monte Carlo particle method [4,6] where we select particles from the initial distribution function and integrate the equations of motion along the characteristics of Eq. (1). We then construct the distribution function $f(V_X, V_Y, X, \tau)$ at time τ from the discrete velocity and space distribution of the particles. For our analysis, the initial distribution is a two-dimensional Maxwellian, uniform in normalized space such that $f(V_X^0, V_Y^0, X^0, 0) = \kappa(2\pi)^{-2}\exp\{-[(V_X^0)^2 + (V_Y^0)^2]/2\}$, where the factor $\kappa(2\pi)^{-1}$ is a normalization constant that reflects the periodicity of the exciting waves and therefore the distribution function in X: $f(V_X, V_Y, -\pi/\kappa, \tau) = f(V_X, V_Y, \pi/\kappa, \tau)$. The characteristics are the solutions to the Hamiltonian

$$H = \frac{1}{2}[P_X^2 + (P_Y - X)^2] + \sum \frac{\varepsilon_i}{\kappa}\cos(\kappa X - \nu_i \tau), \quad (2)$$

where $V_X = P_X$, $V_Y = P_Y - X$, and P_Y is a constant of motion. For our numerical integration, 1000 particles were uniformly spaced in the interval $-\pi/\kappa < X < \pi/\kappa$ with initial velocities randomly selected from the Maxwellian distribution. The equations of motion were solved using a symplectic algorithm [7].

Since the collisionless ion ensemble evolves without thermal equilibration, we followed the convention of Sheng *et al.* [8] in using the average kinetic energy in the direction perpendicular to the magnetic field instead of temperature to gauge ensemble energy: $K(\tau) = \frac{1}{2}\int (V_X^2 + V_Y^2) f(V_X, V_Y, X, \tau) dV_X dV_Y dX$. With our numerical solution for $f(\tau)$, we calculated $K(\tau)$ for a wide range of the wave parameters, ϵ_0, κ, and ν. In each case, this quantity equilibrated to a steady state value K_{eq}, although as predicted from single ion energization considerations [3], the heating time scale decreased with ϵ_0. We show in the top of Fig. 1 plots of the magnitude of K_{eq} for $\epsilon_0 = 5$.

These results are consistent with the range of amplitude values ($0.5 < \epsilon_0 < 10$) we investigated numerically and serve to illustrate, for the first time, that BEWH does produce equal or greater heating than SEWH over a wide range of wave parameters. In order to identify why this is the case and ultimately to see if we can establish BEWH superiority beyond the numerically investigated range, we now use these numerical plots as a guideline and validation in deriving an analytical expression for heating.

To this end, we begin with examining Eq. (3) in greater detail by performing a change of coordinates to an action-angle formulation by means of a generating function of the first kind [9] similar to the one employed by Chia *et al.* [10]: $F_1(X, Y, \theta_1, \theta_2) = 1/2(X - \theta_2)^2\cot\theta_1 + Y\theta_2$. This yields the transformed Hamiltonian:

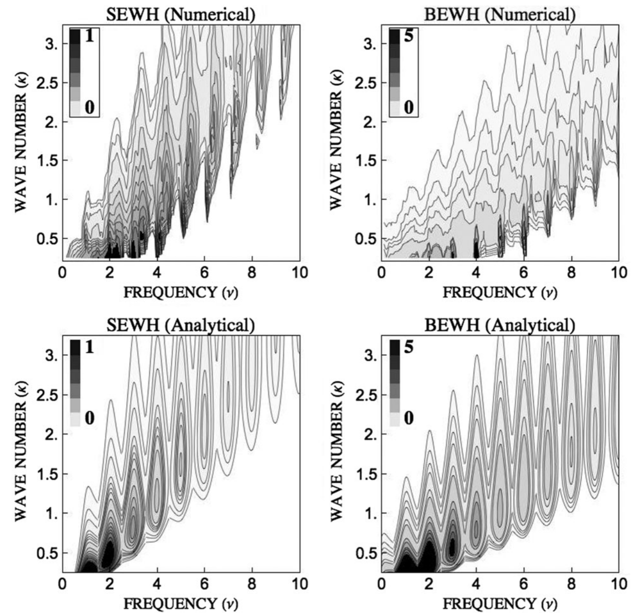

FIG. 1. Numerical and analytical contour plots of equilibrated kinetic energy K_{eq} for both SEWH and BEWH as a function of wave parameters κ and ν: $\epsilon_0 = 5$ in all cases and $\delta = 0.25$ for analytical results. Each set of plots is normalized to the maximum SEWH value in the shown domain.

$$h = I_1 + \sum_{i=1,2}^{2} \frac{\varepsilon_i}{\kappa}\cos(\kappa\sqrt{2I_1}\sin\theta_1 + \kappa\theta_2 - \nu_i\tau), \quad (3)$$

where the coordinate and momenta transformations are given by $X = \sqrt{2I_1}\sin\theta_1 + \theta_2$, $Y = \sqrt{2I_1}\sin\theta_1 - I_2$, $V_X = \dot{X} = \sqrt{2I_1}\cos\theta_1$, and $V_Y = \dot{Y} = -\sqrt{2I_1}\sin\theta_1$. In this case, $I_1 = (V_X^2 + V_Y^2)/2$, θ_1 is the angle of Larmor precision, θ_2 is the position of the guiding center in the X direction, and $-I_2$ is the position of the guiding center in the Y direction. In this formulation I_1 represents the particle kinetic energy such that if we denote the density distribution in action-angle coordinates as $\bar{f}(I_1, I_2, \theta_1, \theta_2, \tau)$, $K(\tau)$ is given by

$$K(\tau) = \int I_1 \bar{f}(I_1, I_2, \theta_1, \theta_2, \tau) dI_1 dI_2 d\theta_1 d\theta_2. \quad (4)$$

Since the ion dynamics are Hamiltonian, we can invoke Liouville's theorem to yield the result consistent with the above Vlasov formulation that $\bar{f}$ is constant along the characteristics $\bar{f}(I_1, I_2, \theta_1, \theta_2, \tau) = \bar{f}(I_1^0, I_2^0, \theta_1^0, \theta_2^0, 0)$, where $\bar{f}(I_1^0, I_2^0, \theta_1^0, \theta_2^0, 0) = \kappa(2\pi)^{-2}e^{-I_1^0}$. Coupled with the conservation of phase space, Eq. (4) becomes

$$K(\tau) = \int \langle I_1(I_1^0, I_2^0, \theta_1^0, \theta_2^0, \tau)\rangle_{\theta^0, I_2^0} e^{-I_1^0} dI_1^0 dI_2^0, \quad (5)$$

where $\langle\cdots\rangle_{\theta^0, I_2^0}$ denotes the average over $0 < I_2^0 < \infty$, $0 < \theta_1^0 < 2\pi$, and $-\pi/\kappa < \theta_2^0 < \pi/\kappa$.

The recasting of Eq. (3) in the coordinates of Eq. (5) now enables us to use Poincaré cross sections for single particle acceleration to approximate the equilibrated value K_{eq}. The Poincaré plots, shown in Fig. 2 and adapted

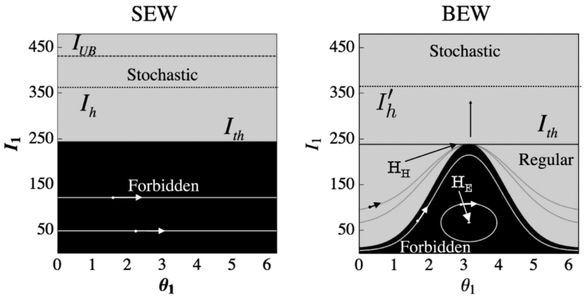

FIG. 2. Figure adapted from Ref. [2]. Poincaré section for the off-resonance case of an ion subject to SEW (left) and BEW (right). The section is a projection into the coordinate plane of ion orbits at fixed time interval τ corresponding to the lowest common period of ν and $\nu + 1$. Typical trajectories for varying initial conditions are shown.

from Spektor and Choueiri [2], are defined for fixed κ and $\tau = N\Delta\tau$, where $N = 0, 1, 2, \ldots$ and $\Delta\tau$ is the least common period of the exciting waves. They depict the trajectories of individual ions in action-angle coordinates for different initial conditions and serve to illustrate the acceleration regions characteristic of each process. On one hand, SEW acceleration is a resonance-broadened process where only ions with initial velocity close to the wave phase velocity, $v_\perp = \omega/k$, are stochastically accelerated. The range of initial actions that satisfy this condition is bounded in phase space by the stochastic threshold [11], $I_{\rm th} = [\nu/\kappa - \sqrt{\varepsilon/\kappa}]^2/2$, and the upper bound, $I_{\rm UB} = \rho_s^2/2 = (4\varepsilon\nu/\kappa)^{4/3}(2/[\pi\kappa])^{2/3}/2$, which is valid provided $I_{\rm UB} \gg (\nu/\kappa)^2 \gg 1$. Ions with initial action outside this resonance zone, i.e., in the forbidden region, are not accelerated. Particle orbits exhibit approximately the same behavior in the forbidden and stochastic regions for BEW acceleration; however, the fundamental aspect of this mechanism—wherein lies its potential for superiority over the SEW process—is that it can accelerate ions outside the resonance zone. This effect occurs in the regular acceleration region, defined by the elliptic point, $H_E = (I_1, \theta_1) = ([\nu/\kappa - \sqrt{\varepsilon/\kappa}]^2/8, \pi)$, and the hyperbolic point, $H_H = ([\nu/\kappa - \sqrt{\varepsilon/\kappa}]^2/2, \pi)$, that intersects the separatrix between regular and forbidden regions [2].

Since θ_1^0 and θ_2^0 are isotropic and the Hamiltonian is independent of I_2^0, we see from Fig. 2 that particles in the SEW forbidden region will have $\langle I_{1({\rm eq})}(I_1^0)\rangle_{\boldsymbol{\theta}^0, I_2^0} = I_1^0$; i.e., the average value of the equilibrated action is constant, while particles in the stochastic region, averaged over initial angles, equilibrate to some value $I_{\rm th} < I_h(\epsilon_0, \nu, \kappa) < I_{\rm UB}$. Assuming $I_{\rm UB} \gg 1$, we see that all ions in an initial Maxwellian with $I_1^0 > I_{\rm th}$ will be in the stochastic region with the remainder in the forbidden region. Therefore, Eq. (3) yields the SEW equilibrated kinetic energy:

$$K_{{\rm eq}(S)} = 1 + \langle I_{\rm eff}\rangle e^{-[\nu/\kappa - \sqrt{\epsilon_0/\kappa}]^2/2}, \tag{6}$$

where $\langle I_{\rm eff}(\epsilon_0, \nu, \kappa)\rangle = I_h - I_{\rm th} - 1$ and $\langle I_1^0\rangle = 1$ corresponds to the initial average action of the ensemble. From this expression, we see that $\langle I_{\rm eff}\rangle$ indicates ions in the stochastic region gain more energy with increasing frequency due to the widening of stochastic phase space. In opposition to this, the term $e^{-[\nu/\kappa - \sqrt{\epsilon_0/\kappa}]^2/2}$ shows that more of the initial distribution falls in the forbidden region with increasing frequency ν.

We use a similar approach as in Eq. (6) to estimate $K_{{\rm eq}(B)}$ for BEWH; however, we define a new effective threshold $I'_{\rm th} = [\nu/\kappa - \sqrt{\epsilon_0/(\sqrt{2}\kappa)}]^2/8$ at the elliptic point to account for the additional ions from the BEW regular acceleration region. We also assume all accelerated ions equilibrate to the BEW value I'_h. The modified Eq. (6) is

$$\langle I\rangle_{\rm eq} = 1 + \langle I'_{\rm eff}\rangle e^{-\{\nu/\kappa - (\epsilon_0/[\kappa\sqrt{2}])^{1/2}\}^2/8}, \tag{7}$$

where $\langle I'_{\rm eff}\rangle = I'_h - I'_{\rm th} - 1$.

Both Eqs. (6) and (7) are based on the analysis of a Poincaré cross section valid only for off-resonance frequencies ($\nu \neq$ integer). However, an examination of on-resonance acceleration for both SEW [12,13] and BEW [2] showed that there is little change in the boundary of the stochastic region. The significant difference caused by on-resonance effects is the appearance of a web structure in the stochastic region that leads to acceleration beyond the maximum, $I_{\rm UB}$. This suggests that Eqs. (6) and (7) can be universally applied provided the on-resonance effects are folded into $I_{\rm eff}$ and $I'_{\rm eff}$.

In order to find these terms, we need to evaluate Eq. (5) analytically. Since the nonlinearity of Eq. (3) precludes a closed form solution for $I_1(I_1^0, I_2^0, \theta_1^0, \theta_2^0, \tau)$, we invoke the results of Lie transform theory [14]. For small amplitude ($\epsilon_0 < 1$) and an appropriate Hamiltonian, this yields $\langle I_1(\tau)_2\rangle_{\boldsymbol{\theta}^0} = \langle I_1^0\rangle_{\boldsymbol{\theta}^0} + \frac{1}{2}\partial_{I_1^0}\langle[(\partial_{\theta_0^1}\Delta w_1)(\partial_{\theta_0^1}\Delta w_1)]\rangle_{\boldsymbol{\theta}^0}$, where $\Delta w_1 = -\int_0^\tau d\tau H_1$, H_1 is the first order term in ε_i from Eq. (3), the subscript 2 denotes second-order quantities, and the integration is performed over the orbits in phase space pertaining to the solution of the unperturbed Hamiltonian, $H_0 = I_1$. This expression is valid provided the generating functions up to second order from the Lie transform of Eq. (3) are periodic with respect to phase angle and the transformed Hamiltonian (denoted by K in Ref. [14]) is independent of the phase angles. These criteria are satisfied for small ϵ_0 and in the off-resonance case, $\nu \neq$ integer, such that Eq. (5) becomes

$$K(\tau)_2 = \sum_{m=-\infty i,j=1,2}^{\infty} \frac{\varepsilon_i\varepsilon_j}{\kappa^2}\int_0^\infty \frac{\partial}{\partial I_1^0}[m^2 J_m(\kappa\sqrt{2I_1^0})^2]e^{-I_1^0}dI_1^0 \frac{\cos[(\nu_i - \nu_j)\tau] - \cos[(m - \nu_i)\tau] - \cos[(m - \nu_j)\tau] + 1}{4(\nu_i - m)(\nu_j - m)} + 1,$$

which is a time dependent expression for the average ion kinetic energy where $J_m(\cdots)$ denotes the mth order Bessel function of the first kind. Averaging over time yields

$$K_{\rm eq} = 1 + \frac{1}{2}\sum_{i=1,2}\frac{1}{(\|\nu\| - \nu)^2 + \delta^2}\left(\frac{\varepsilon_i}{\kappa}\right)^2\int_0^\infty \frac{\partial}{\partial I_0} \times [\|\nu_i\|^2 J_{\|\nu_i\|}(\kappa\sqrt{2I_0})^2]e^{-I_0}dI_0,$$

the equilibrated average ion kinetic energy, where we have dropped the second-order subscript, simplified the infinite summation over m by retaining only the dominant $m = \|\nu_i\|$ terms ($\|\cdots\|$ denotes the nearest integer function), and introduced the constant $\delta^2 \ll 1$ in order to remove the nonphysical singularity at on resonance ($\nu = \|\nu\|$). This expression is a special case of Weber's second exponential integral, which yields (Watson [15], p. 395)

$$K_{\rm eq} = 1 + \frac{e^{-\kappa^2}}{2[(\|\nu\| - \nu)^2 + \delta^2]}\sum_{i=1,2}\left(\frac{\varepsilon_i\|\nu_i\|}{\kappa}\right)^2 I_{\|\nu_i\|}(\kappa^2),$$

where $I_{\|\nu_i\|}(\kappa^2)$ is the modified Bessel function of the first kind. Furthermore, in the $\nu/\kappa > 1$ limit, we find

$$K_{\rm eq} = 1 + \frac{1}{(\|\nu\| - \nu)^2 + \delta^2}\sum_{i=1,2}\left(\frac{\varepsilon_i\|\nu_i\|}{8\pi\kappa^{3/2}}\right)^2 e^{-(1/2)(\|\nu_i\|/\kappa)^2}. \quad (8)$$

From this result, we see that the exponential terms in Eqs. (6) and (8) are almost identical where $\|\nu_i\|$ has replaced ν_i, and in our small ϵ_0 analysis the amplitude dependent term $\sqrt{\epsilon_0/\kappa}$ is absent. This implies to second order $\langle I_{\rm eff}\rangle \approx (\|\nu\| - \nu)^2 + \delta^2)^{-1}(\frac{\epsilon_0\|\nu\|}{8\pi\kappa^{3/2}})^2$, which with Eqs. (6) and (8) yields an approximation for SEWH:

$$K_{{\rm eq}(S)} = 1 + \frac{1}{(\|\nu\| - \nu)^2 + \delta^2}\left(\frac{\epsilon_0\|\nu\|}{8\pi\kappa^{3/2}}\right)^2 e^{-(\nu/\kappa - \sqrt{\epsilon_0/\kappa})^2/2}. \quad (9)$$

This expression was derived under the assumption $\epsilon_0 < 1$; however, we have found it to be consistent over our numerically investigated range $0.5 < \epsilon_0 < 10$. This is illustrated by Fig. 1 where we see the plot of Eq. (9) corresponds quite well to the numerically indicated data.

Turning to the case of BEWH, it is apparent from comparing Eqs. (7) and (8) that our second-order expression for $K_{{\rm eq}(B)}$ is not accurate. The discrepancy in the exponential terms indicates that Eq. (8) fails to capture the effect of the regular acceleration region shown in the BEW Poincaré cross section. Indeed, our analysis reveals that the BEW effect only appears at higher order in ϵ_0—an observation consistent with single wave findings [10]. Since the analytical expression for heating becomes prohibitively complicated beyond second order, we make the simplifying assumption that $\langle I'_{\rm eff}\rangle = \langle I_{\rm eff}\rangle$, which is justified by single ion studies that have indicated the stochastic region remains qualitatively the same for both BEW and SEW acceleration [2]. This simplification, combined with the exponential of Eq. (7) that exclusively represents the regular acceleration, thus yield an expression for BEWH:

$$K_{{\rm eq}(B)} = 1 + \frac{1}{(\|\nu\| - \nu)^2 + \delta^2} \times \left(\frac{\epsilon_0\|\nu\|}{8\pi\kappa^{3/2}}\right)^2 e^{-\{\nu/\kappa - (\epsilon_0/[\kappa\sqrt{2}])^{1/2}\}^2/8}. \quad (10)$$

By comparing Eq. (10) with numerical results for the investigated range of ϵ_0 in Fig. 1, we see this expression successfully describes BEWH without expanding to fourth order. Equation (10) thus generalizes our numerical results and permits a direct analytical comparison between SEWH and BEWH for arbitrary wave parameters: for all positive values of κ, ν, and ϵ_0, BEWH is always greater than or equal to SEWH. Furthermore, the good agreement of our results with numerical work provides strong support for our supposition that the major difference between BEWH and SEWH lies in the fraction of particles subject to acceleration [as indicated by the different exponential terms in Eqs. (9) and (10)]. We thus confirm that BEWH's superiority stems from its ability to energize more of an ion ensemble concurrently for a given wave energy density.

Finally, the explicit dependence of BEWH on wave parameters allows us to reincorporate an element of self-consistency by substituting a dispersion relation, $D(\kappa, \nu) = 0$, into Eq. (10). In this way, we can predict, given actual experimental constraints and without resorting to a simulation, the resulting heating.

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