Fundamental Limit of Self-Field MPD Thruster Efficiency

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A fundamental upper limit, $\hat{\eta}$, for the thrust efficiency of self-field magnetolpasmadynamic thrusters (MPDTs) is derived from the generalized Ohm's law and the minimization of the volume integral of the square of the current density, which controls dissipation in the MPDT. It is found that $\hat{\eta} \simeq 1/(1+4/R_{m_{ci}}\xi^4)$, where ξ is the MPDT scaling number (the total current normalized by the current at which an equipartition of power occurs between thrust and ionization), and $R_{m_{ci}}$ is the magnetic Reynolds number evaluated at $\xi = 1$.

I. Motivation

It would obviously be useful to have a measure of the headroom magnetolpasmadynamic thruster (MPDT) performance has for improvement through judicious design before it reaches a fundamental limit. To this end we seek an expression for the upper limit of MPDT thrust efficiency that does not explicitly depend on the dimensional parameters related the design of the thruster and its operation.

II. Derivation

A. Fundamental MPDT Power Balance

We start with the generalized Ohm's law[1], which relates the current density, $\boldsymbol{j}$, the magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}$, in the form that is more useful for analyzing the MPDT:

$$\boldsymbol{E}' = \frac{1}{\sigma}\left[\boldsymbol{j} + \Omega_e \frac{\boldsymbol{j}\times\boldsymbol{B}}{B} + s\frac{\boldsymbol{B}\times(\boldsymbol{j}\times\boldsymbol{B})}{B^2}\right], \tag{1}$$

where $\sigma = n_e e^2/m_e \bar{\nu}_e$ is the scalar conductivity, n_e the electron density, m_e the electron mass, e the universal charge, $\bar{\nu}_{eH}$ the electron-heavy species momentum-averaged collision frequency, $\Omega_s = \omega_{cs}/\bar{\nu}_s$ the electron Hall parameter for species s, $\omega_{cs} = eB/m_s$ the electron cyclotron frequency, $s = (\rho_n/\rho)^2\Omega_e\Omega_i$ the ion slip factor, ρ the total mass density, and ρ_n the neutral mass density. For the MPDT, the high collisionality and high ionization fraction[2] insure that the ion slip term is negligible, and we are left with

$$\boldsymbol{j} = \sigma\boldsymbol{E}' - \Omega_e\frac{\boldsymbol{j}\times\boldsymbol{B}}{B}.$$

Since our aim is to study the energetics of the MPDT, we take the dot product of the above equation with $\boldsymbol{j}$, and note that $\boldsymbol{j}\cdot\boldsymbol{j}\times\boldsymbol{B} = 0$, to get

$$\frac{j^2}{\sigma} = \boldsymbol{j}\cdot\boldsymbol{E}' = \boldsymbol{j}\cdot\boldsymbol{E} - \boldsymbol{u}\cdot\boldsymbol{j}\times\boldsymbol{B}, \tag{2}$$

where, in the second equality, we have reverted to the electric field in the laboratory frame. We note the convenient fact that, with the Hall term disappearing, we only have to deal with the *scalar* conductivity, without losing the generality

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This paper is dedicated to Professor Mariano Andrenucci.

that conduction in the plasma can be anisotropic. Taking the integral over the volume, $\mathcal{V}$, occupied by the discharge, we have

$$\int_{\mathcal{V}} \boldsymbol{j} \cdot \boldsymbol{E}\, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{u} \cdot \boldsymbol{j} \times \boldsymbol{B}\, d\mathcal{V} + \int_{\mathcal{V}} \frac{j^2}{\sigma}\, d\mathcal{V}. \tag{3}$$

We now make two phenomenological arguments to express the first two integrals in forms that will help us in our ultimate derivation of $\hat{\eta}$:

1) The first integral, $\int_{\mathcal{V}} \boldsymbol{j} \cdot \boldsymbol{E}\, d\mathcal{V}$, represents the power input into the plasma excluding the electrode sheaths, and can be expressed as $J(V - V_e)$, where J is the total current through the thruster, and $V_e = V_c + V_a$ is the sum of the cathode and anode sheath voltages.
2) The second integral, $\int_{\mathcal{V}} \boldsymbol{u} \cdot \boldsymbol{j} \times \boldsymbol{B}\, d\mathcal{V}$, represents the work done by the Lorentz force, which acts to accelerate the plasma and produce thrust. We expect it to be linearly related to the thrust power $\dot{m}u_e^2/2$, where $u_e = T/\dot{m}$ is the exhaust velocity, T is the thrust, and $\dot{m}$ is the mass flow rate. We will therefore take the integral to be $\alpha T^2/2\dot{m}$, where α is the linear factor.

We will have much to say further below about the last integral, $\int_{\mathcal{V}} j^2/\sigma\, d\mathcal{V}$, which represents Joule heating.

With these identifications, Eq. 3 then becomes

$$J(V - V_e) - \alpha \frac{T^2}{2\dot{m}} = \int_{\mathcal{V}} \frac{j^2}{\sigma}\, d\mathcal{V}, \tag{4}$$

and is central to our derivation of the MPDT efficiency limit. It states a power balance that is fundamental to the MPDT.

To underline the fundamental nature of this expression, we note that Villani[3] arrived at the exact equation above after a very detailed derivation involving the MHD continuity, momentum, and energy equations, the generalized Ohn's law, Maxwell's equations, the second law of thermodynamics, dimensional analysis, volume integration, and the divergence theorem[a]. The two phenomenological arguments we made above to quickly arrive at the same result can be shown to reflect fundamental relations that he derived from first principles, as discussed in the Appendix.

Villani[3] also extensively verified this balance experimentally via detailed experimental measurements with 5 thrusters geometries and a wide range of operation conditions over which $\int_{\mathcal{V}} j^2/\sigma\, d\mathcal{V}$ varied by a factor of 370. Moreover, he showed that the conductivity varies by no more than a factor of 1.6, while the measured value of $\boldsymbol{j}^2$ varies by more than a factor of 100 over the discharge volume. This allows pulling the conductivity out of the volume integral. Finally, he showed that α is in the range 1.15-1.4. Therefore, the power balance (with α set to 1) becomes

$$J(V - V_e) = \frac{T^2}{2\dot{m}} + \frac{1}{\sigma} \int_{\mathcal{V}} j^2\, d\mathcal{V}. \tag{5}$$

B. MPDT Efficiency, Thrust, and the MPDT Scaling Number

Using the basic definition of thrust efficiency, η, of electric rockets, and the above MPDT power balance, we have

$$\eta \equiv \frac{T^2}{2\dot{m}VJ} = \frac{1}{1 + \dfrac{2\dot{m}}{\sigma T^2} \displaystyle\int_{\mathcal{V}} j^2\, d\mathcal{V} + \beta_e}, \tag{6}$$

where $\beta_e = V_e J/(T^2/2\dot{m})$ is the ratio of the power dissipated in the electrode sheaths to the thrust power, and quickly becomes less than 1 as the thruster is operated at the high current levels, nearing the "onset condition", required for high efficiency[4–7]. Setting $\beta_e = 0$ is furthermore justified by our seeking an upper limit for η.

It is clear from the above expression that the volume integral represents the main source of inefficiency. We shall call it the "dissipation integral", I_D

$$I_D \equiv \int_{\mathcal{V}} j^2\, d\mathcal{V},$$

and seek to find its fundamental lower limit, in order to find the upper limit of η. Before we do so, we turn our attention to the thrust.

It is widely established that the thrust of the MPDT when operated in the high current regime (defined as $\xi \geq 1$ in terms of the MPDT scaling number we shall define shortly) is overwhelmingly due to its electromagnetic component

[a] Villani's work was documented in a PhD thesis[3] but was never published in a conference or journal paper.

(see, for instance Ref. [8], and Table A-1 of Ref. [3], where the ratio of the electrothermal to electromagnetic components of thrust, T_{et}/T_{em} is found to range between 3.9 and 1.7% in the high current regime.) Therefore, we take $T \simeq T_{em}$.

We have addressed the electromagnetic thrust of the MPDT in Ref [9]. Specifically, the "thrust coefficient", C_T, defined as

$$C_T \equiv \frac{4\pi}{\mu_o} \frac{T}{J^2}, \tag{7}$$

and which is of order of unity, and can be measured accurately with a thrust stand and a current probe, was found to obey the following scaling with ξ

$$C_T = \frac{\nu}{\xi^4} + \ln\left(\frac{r_a}{r_c} + \xi^2\right), \tag{8}$$

where ν is a non-dimensional parameter that depends only the mass flow rate and is of order 10^{-2}, r_a/r_c is the ratio of anode to cathode radii, and ξ is the "MPDT scaling number" defined below. This scaling of C_T was verified extensively with thrust stand measurements taken with argon, krypton and xenon propellants. At $\xi \simeq 1$, $C_T \simeq \ln(r_a/r_c)$ due to the vanishing first term and the attenuation of the log function. Therefore, we have $T \simeq T_{em} = bJ^2$, $u_{ex} = T/\dot{m} = bJ^2/\dot{m}$, where, to a good approximation, $b = (\mu_o/4\pi)\ln(r_a/r_c)$ depends only on the thruster's geometry, and represents its inductance per unit length.

The "MPDT scaling number" (whose derivation was first given in Ref, [9], and which we summarize here for convenience) is tied to the concept of *nominal operation*, which is the operation point at which the directed kinetic power in the exhaust (associated with thrust) is equal to the power associated with the ionization sink when the entire mass flow rate is at the first ionization potential, ϵ_i:

$$\frac{1}{2} T u_{ex} = \dot{m}\frac{\epsilon_i}{M}. \tag{9}$$

Here M is the atomic mass, and ϵ_i is the first ionization potential. This equipartition is substantiated by the finding that ionization is a significant power sink, whose magnitude, at nominal thruster operation conditions, is of the same order as that of the power associated with acceleration[3], and by spectroscopic measurements that show that less than 10% of the power tied in ionization is recovered in the exhaust, and that as much as 85% of the internal energy is in ionization[8] .

Defining "nominal thruster operation" in terms of this power equipartition, we have

$$\left(\frac{\mu_o}{4\pi}\ln\frac{r_a}{r_c}\right)^2 J^4 = \dot{m}^2 u_{ci}^2, \tag{10}$$

where u_{ci} is defined as

$$u_{ci} \equiv \left(\frac{2\epsilon_i}{M}\right)^{1/2}, \tag{11}$$

and is known as the critical ionization velocity. For xenon, krypton, argon, and lithium, u_{ci} is 4.22, 5.68, 8.72 and 12.24 km/s, respectively. The current at which the nominal operation condition is reached is called the critical ionization current, J_{ci}. From the above equation we have,

$$J_{ci} = \left(\frac{\dot{m} u_{ci}}{b}\right)^{1/2}. \tag{12}$$

This characteristic current is used to normalize the total current leading us to the dimensionless parameter ξ, which we call the MPDT scaling number:

$$\xi \equiv \frac{J}{J_{ci}}. \tag{13}$$

More explicitly in terms of the controllable parameters of thruster operation,

$$\xi = J \Bigg/ \left[\frac{\dot{m}^{1/2}\,(2\epsilon_i/M)^{1/4}}{\left(\frac{\mu_o}{4\pi}\ln\frac{r_a}{r_c}\right)^{1/2}}\right]. \tag{14}$$

When the MPDT scaling number is equal to unity, the thruster is said to be at its nominal operation condition, with a nominal specific impulse $I_{sp} = u_{ci}/g_o$, irrespective of the input power level.

Noting that $T = \dot{m}u_{ci}\xi^2$, we can now express the efficiency in Eqn. 6 as a function of ξ:

$$\eta \simeq \frac{1}{1 + \dfrac{2}{\sigma \dot{m} u_{ci}^2 \xi^4} \displaystyle\int_{\mathcal{V}} j^2 \, d\mathcal{V}}. \tag{15}$$

To find the upper limit of the efficiency we now seek the lower limit of the dissipation integral.

C. The Lower Limit of the Dissipation Integral and the Upper Limit of Efficiency

We use the Lagrange multiplier method to find the lower limit of I_D. From Eq. 2 we see that minimizing $\int_{\mathcal{V}} j^2 d\mathcal{V}$ is equivalent to minimizing $\sigma \int_{\mathcal{V}} \boldsymbol{j} \cdot \boldsymbol{E}' \, d\mathcal{V}$, subject to the following two constraints

$$\nabla \cdot \boldsymbol{j} = 0, \tag{16}$$

$$\nabla \cdot \boldsymbol{E}' = 0, \tag{17}$$

and this leads us to write the functional

$$\mathcal{F}(\boldsymbol{x}, \boldsymbol{j}, \boldsymbol{E}') = \int_{\mathcal{V}} \left(\boldsymbol{j} \cdot \boldsymbol{E}' + \lambda_1 \nabla \cdot \boldsymbol{j} + \lambda_2 \nabla \cdot \boldsymbol{E}' \right) \, d\mathcal{V},$$

where $\boldsymbol{x}$ represents the spatial variables, and λ_1, λ_2 are the two Lagrange multipliers. Hildebrand [10] describes the method of minimizing integrals of the form

$$\mathcal{F}(x, u, w) = \int F(x, u, u', w, w') \, dx,$$

(where the prime, in this context, denotes a spatial derivative) using Lagrange multipliers, and shows that the solution is obtained by solving the associated Euler equations:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0, \tag{18}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right) - \frac{\partial F}{\partial w} = 0. \tag{19}$$

Generalizing the spatial differential operator to vector form, $d/dx \to \nabla \cdot$, and applying the method to our particular problem, where $F = \boldsymbol{j} \cdot \boldsymbol{E}' + \lambda_1 \nabla \cdot \boldsymbol{j} + \lambda_2 \nabla \cdot \boldsymbol{E}'$; $u = \boldsymbol{j}$; $w = \boldsymbol{E}'$; $u' = \lambda_1 \nabla \cdot \boldsymbol{j}$; and $w' = \lambda_2 \nabla \cdot \boldsymbol{E}'$ we find

$$\boldsymbol{E}' = \nabla \lambda_1, \tag{20}$$

$$\boldsymbol{j} = \nabla \lambda_2. \tag{21}$$

The first result identifies λ_1 as the negative of the electric potential ϕ_E, while the second result, of more interest to our problem, implies that also the current density can be expressed in terms of a potential $\lambda_2 = -\phi_j$. By virtue of Eq. 16, the current density potential satisfies Laplace's equation $\nabla^2 \phi_j = 0$. Since $\nabla \cdot \boldsymbol{j} = -\nabla \phi_J$ it follows that

$$\nabla \times \boldsymbol{j} = 0 \quad \text{(when } I_D \text{ is minimum)}. \tag{22}$$

In other words, the lower limit of our dissipation integral is reached when the current pattern is irrotational, which would be the case of uniform current conduction throughout the discharge[b].

Knowing the nature of the current pattern that minimizes the dissipation integral, we proceed to express its minimum value, $I_{D_{min}}$, in terms of the relevant operation and geometric parameters of a cylindrical self-field MPDT operating at

[b]Villani[3] makes the same statement but provides no proof. Instead he cites, erroneously, a math textbook where no such proof exists, thus our explicit derivation. Also, it is straightforward to show that the same result, namely $\nabla \times \boldsymbol{j} = 0$, is obtained in the more general case where the conductivity is both a tensor and spatially varying.

a total current J. For that we take $\boldsymbol{j} = \hat{\boldsymbol{r}} j_r$, then $j_r = J/(2\pi l_j r)$, where l_j is the axial extent of the current pattern in the cylindrical discharge, and carry out the volume integration in cylindrical coordinates to get

$$I_{D_{min}} = \frac{J^2}{l_j^2 (2\pi)^2} \int_{r_a}^{r_c} \int_0^{2\pi} \int_0^{l_j} \frac{dr}{r}\, d\theta\, dz = \frac{J^2}{2\pi l_j} \ln \frac{r_a}{r_c}. \tag{23}$$

Using the above expression for $I_{D_{min}}$ in Eq. (15) to get the upper limit for the efficiency $\hat{\eta}$, we find

$$\widehat{\eta} = \frac{1}{1 + \dfrac{4}{\sigma \mu_0 u_{ci} l_j \xi^2}}. \tag{24}$$

where we recognize $\sigma \mu_0 u_{ci} l_j$ as the magnetic Reynolds number evaluated at $u_e = u_{ci}$ (that is $\xi = 1$),

$$R_{m_{ci}} \equiv \sigma \mu_0 u_{ci} l_j, \tag{25}$$

which we use to cast $\hat{\eta}$ in its final form

$$\hat{\eta} = \frac{1}{1 + \dfrac{4}{R_{m_{ci}} \xi^2}}. \tag{26}$$

III. Discussion and Application

The physical picture reflected in the above deceptively simple expression is that of a competition between electromagnetic thrust power, which scales with ξ^4, and Joule heating, which is invested in the ionization power sink, and is effectively irrecoverable as thrust, and which scales[c] with ξ^2. This input power is equipartitioned between these two sinks at the nominal condition where, by definition, $\xi = 1$.

If ξ is decreased, $I_D \to I_{D_{min}}$ and the current density pattern becomes more irrotational, which is a hallmark of low dissipation, but thrust efficiency actually decreases as thrust power vanishes quicker. If ξ is increased, thrust power eventually takes over dissipation, and η increases. The headroom the efficiency has to increase, at a given value of ξ for a given MPDT, is manifested by how far the current density pattern is from being irrotational. Such headroom can be used to increase performance through judicious design (e.g. lengthening the cathode[3], or contouring the electrodes[11]) and optimized schemes (e.g. distributed mass injection[12]).

Since the extent to which the current density pattern is drawn axially (and thus made less irrotational) with the flow, is represented by the magnetic Reynolds number, that number scales the headroom. Indeed, the value of R_m at $\xi = 1$, becomes the sole measure of that headroom at the nominal operation condition,

$$\hat{\eta}_{ci} \equiv \hat{\eta}(\xi = 1) = \frac{R_{m_{ci}}}{4 + R_{m_{ci}}}, \tag{27}$$

which is 20%, at $R_{m_{ci}} = 1$, and 71% at $R_{m_{ci}} = 10$. This means there is more headroom to increase the efficiency by judicious design and operation schemes (the realm of practical MPDT research) at higher $R_{m_{ci}}$ – that is with higher conductivity and higher values of u_{ci}.

A felicitous feature of Eq. 26, aside from its simplicity, is that $\hat{\eta}(\xi^2)$ depends on a single parameter (R_m) that is dimensionless, and which needs to be evaluated at only one condition (the nominal operation condition, $\xi = 1$), using the characteristic velocity u_{ci}, which in turn depends only on the propellant's atomic mass and ionization potential.

The axial extent of the discharge, l_j, which corresponds to the length of the cylindrical conduction region when I_D is minimum, is essentially the cathode length. For operational requirements (e.g. allowable head load, required lifetime) and system limitations (size and weight) that length is on the order of 10^{-1} m (typically 20 cm).

MPDTs are always operated in the more efficient high-current ($\xi \geq 1$) regime. There is now clear experimental evidence that the injected gas in the MPDT is abruptly and effectively ionized by a thin ionization front located far upstream in the discharge and, when $\xi \geq 1$, that the same high level of ionization is maintained throughout the discharge. This justifies using the Spitzer-Härm formula to calculate the conductivity[1]:

$$\sigma \simeq 1.53 \times 10^{-2} \frac{T_e^{3/2}}{\ln \Lambda} \ \text{mhos/m},$$

[c] While the ionization sink *power* scales with ξ^2, the ionization *fraction* is shown in Ref. [2], through spectroscopic measurements, to scale with ξ.

where Λ is the plasma parameter and is given by

$$\Lambda = 1.24 \times 10^7 \left(\frac{T^3}{n_e}\right)^{1/2} \quad \text{(with } T \text{ in K, and } n_e \text{ in m}^{-3}\text{)}.$$

Finally, it must be pointed out that there is another practical limit to increasing the efficiency due to the so-called "onset condition," which is reached when the current is raised enough for a given mass flow rate, and at which significant erosion of the anode occurs accompanied by severe hash in the voltage signal[13, 14]. We denote this condition by ξ^*.

As an illustrative example of the application of Eq. 26, we show in Table 1 calculations of $\hat{\eta}$ for the cases of an MPDT operated with four monatomic propellants: argon, xenon, krypton and lithium. We take l_j=20 cm, and assume that the conductivity, $\sigma \simeq 5128$ mhos/m, calculated from the above expressions for the experimental conditions of [3] and verified experimentally by Villani, does not vary between the four plasmas (an assumption that holds well if the plasmas are fully ionized and have the same $\ln \Lambda$).

	Xe	Kr	Ar	Li
u_{ci}	4.22	5.68	8.72	12.24
$R_{m_{ci}}$	5.4	7.32	11.24	15.78
$\hat{\eta}(\xi = 1)$	57.4	64.7	73.7	79.8
$\hat{\eta}(\xi = \xi^*)$	72.6	78.2	84.6	88.5

Table 1 Calculations of the limits of MPDT efficiency for four propellants. ($l_j = 20$ **cm,** $\xi^* = 1.4$**).**

Strictly speaking, the application of our expression for $\hat{\eta}$ is limited to monoatomic propellants since the definition of ξ involves a single mass and a single ionization potential. The extent to which the above theory can be amended or evolved to be applicable for molecular propellant remains to be explored.

Appendix

Villani[3] derived the following relation from first principles:

$$\nabla \cdot (\rho \boldsymbol{u} h_0) = -\nabla \cdot (\boldsymbol{j} \phi_E), \tag{28}$$

where $h_0 = h + u^2/2$ is the stagnation enthalpy, $h = \epsilon + p/\rho$ the enthalpy, and ϵ the specific energy. Then by integrating this identity over the discharge volume he found, for the left side of the above equation,

$$\int_{\mathcal{V}} \nabla \cdot (\rho \boldsymbol{u} h_0)\, d\mathcal{V} = \dot{m}\overline{h_0} = \dot{m}\frac{\overline{u^2}}{2} + \dot{m}\epsilon,$$

where the bars indicate mass-averaged quantities, and for the right side,

$$-\int_{\mathcal{V}} \nabla \cdot (\boldsymbol{j} \phi_E)\, d\mathcal{V} = J(V - V_E).$$

The above two expressions reduce Eq. 28 to the power balance

$$J(V - V_E) = \dot{m}\frac{\overline{u^2}}{2} + \dot{m}\epsilon. \tag{29}$$

Then by combining the generalized Ohm's (Eq. 2) with the MHD energy equation

$$\rho \boldsymbol{u} \cdot \nabla \epsilon = \boldsymbol{j} \cdot \boldsymbol{E} - \boldsymbol{u} \cdot \boldsymbol{j} \times \boldsymbol{B},$$

from which all other terms (viscous dissipation, heat flux, and $pd\mathcal{V}$ work) have been justifiably dropped through a dimensional analysis, the following equality is found

$$\rho \boldsymbol{u} \cdot \nabla \epsilon = \frac{j^2}{\sigma},$$

which, as before, through a volume integration and the use of the divergence theorem yields

$$\dot{m}\bar{\epsilon} = \int_{\mathcal{V}} \frac{j^2}{\sigma}\, d\mathcal{V}.$$

Finally, through a similar use of volume integration, the divergence theorem, Villani shows that the remaining term in Eq. 29, can be expressed as

$$\frac{\dot{m}\overline{u^2}}{2} = \frac{\dot{m}u_e^2}{2} = \alpha\frac{T^2}{2\dot{m}},$$

leading to the MPDT fundamental power balance we derived, Eq. (4).

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