



Efficiency of Plasma Heating with Beating Electrostatic Waves

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A one-dimensional efficiency model is derived for the heating of a uniformly magnetized plasma with beating electrostatic waves (BEW). Due to the non-resonant nature of this process, it is believed to offer improvements over existing resonant schemes for plasma heating in electric propulsion applications. A simplified energy transport equation with a Fokker-Planck diffusion operator for the interaction of the BEW with a magnetized plasma is used to predict the efficiency of heating in a rectilinear geometry for waves with phase velocities larger than the ion thermal velocity. An explicit calculation for efficiency is performed for the case where the BEW consist of two electrostatic ion cyclotron waves. The resulting expression matches the observed heating efficiency in a BEW laboratory experiment to within an order of magnitude, and the low efficiency values observed in this laboratory experiment are shown to be the result of an unfavorable set of plasma parameters where the ratio of wave phase velocity to ion thermal velocity is exceptionally high. In order to examine the efficacy of BEWH for an electrothermal propulsion concept, the plasma parameter space of a typical radiofrequency plasma propulsion concept with a lower ratio of wave to ion velocity is investigated. It is shown that under these conditions, BEW heating is capable of reaching high efficiency levels.

I. Introduction

The radiofrequency (RF) heating of plasma is an integral component for investigations in atmospheric plasma dynamics and fusion research as well as for applications ranging from industrial processes to electric propulsion. In this last respect, the RF heating of plasmas is particularly attractive for magnetized, electrothermal propulsion concepts. Since such thrusters depend on the conversion of randomized kinetic energy from a plasma heating element to directed exhaust energy, the lifetime of these propulsion systems is severely limited by the erosion of the electrodes that are exposed to the plasma. The advantage of employing RF heating in these systems lies in its ability to couple power into the propellant from an antenna placed outside the plasma. This permits considerable power transfer to the propellant while minimizing thruster component erosion.

The success of this scheme for electric propulsion is inherently limited by the efficiency of the chosen RF heating method. The most popular RF scheme currently employed in electric propulsion is ion cyclotron resonance heating (ICRH) in a magnetized plasma.^{1,2} However, while this process has been shown to couple relatively well to the ions, the efficiency of power conversion in this scheme is limited by its inherent dependence on resonance conditions. That is, only a small fraction of the particles in the plasma with appropriate velocity and magnetic field will interact with the wave.

In light of this constraint, it has been proposed that the beating electrostatic wave heating (BEWH) of a plasma may offer improved efficiency over existing resonant heating techniques.³ The reason for this stems from the unique ability of BEW to *non-resonantly* energize ions.⁴⁻⁶ While the interaction of a perpendicularly-propagating single electrostatic wave (SEW) with a plasma is constrained to those magnetized ions with perpendicular velocity, v_{\perp} , close to the wave phase velocity, v_{ϕ} , such that $v_{\phi} \sim v_{\perp}$, the BEW process employs two electrostatic waves propagating perpendicularly to an ambient magnetic field with frequencies $\omega_{1,2}$ that satisfy the so-called beating criterion $\omega_2 - \omega_1 = n\omega_{ci}$ where ω_{ci} is in the ion cyclotron frequency, and n is an integer. These waves produce an electric field amplitude with an envelope that beats at

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the ion cyclotron frequency. This slow oscillation allows coupling with ions whose velocities are significantly lower than the phase velocity of the individual waves—thereby leading to non-resonant acceleration.

The BEW mechanism has been examined in great detail through single particle analysis.^{4–9} Ram *et al* first proposed the process to explain observations of non-resonant ion acceleration in the ionosphere.⁸ Benisti *et al*^{5,6} subsequently derived the necessary condition for ion acceleration—the aforementioned beating criterion—then Spektor and Choueiri expanded on these results by showing that there are additional necessary and sufficient conditions for an ion to undergo non-resonant acceleration.⁴ Jorns and Choueiri were the first to bridge the gap from single particle analysis to plasma heating by deriving an expression for the equilibrated energy of an initially thermalized ensemble subject to collisionless BEWH.³ In that work, it was concurrently demonstrated that for equal energy densities, the BEW process outperformed the well-studied process of resonant heating with SEW. Small-scale laboratory investigations in the Beating Waves Experiment (BW X I) and Beating Waves Experiment II (BW X II) have subsequently confirmed that BEWH does occur and have demonstrated the theoretically anticipated superiority of BEWH to resonant SEW heating for comparable input powers.^{10,11} A method for achieving direct ion acceleration with BEW has also been proposed for propulsion applications, but this is a subject outside the domain of electrothermal propulsion.¹²

In light of these studies, it is apparent that BEWH is an attractive candidate for a heating stage in an electrothermal plasma propulsion concept. However, to date, there has been no experimental measurement or analytical calculation of the *efficiency* of heating using BEWH. Thus, while we know that the process can heat a plasma, we do not have a metric for estimating its performance in an electric propulsion concept or for comparing it to other, non-electrostatic heating schemes such as ICRH. In order to address this shortcoming in the characterization of BEWH, we have set as the goal of this investigation the derivation of a first order expression for the heating efficiency of BEWH.

To this end, this paper is organized in the following way. In the first section, we introduce a simplified expression for heating efficiency in terms of the incident wave power. In the second section, we follow the treatment of Karney¹³ to derive an expression for the power of BEWH using a Fokker-Planck formulation. In the third section, we arrive at a simplified model for the heating efficiency of BEWH when electrostatic ion cyclotron waves (EICW) are employed and compare the calculated efficiencies with estimated efficiencies from BW X II. In the fourth section, we calculate the predicted efficiency of BEWH with the efficiency of ICRH in an electric propulsion system in order to illustrate the efficacy of the BEW process. In the final section, we discuss the validity of our results and implications for future applications of this mechanism to electric propulsion.

II. Heating Efficiency

The commonly accepted definition for the efficiency of an RF heating scheme is¹⁴

$$\eta = \frac{n_i V \Delta T_i}{\tau_c P_{rf}} \quad (1)$$

where n_i denotes the ion density, V is the heating volume, $\Delta T_i = T_i - T_{i0}$ is the increase in the average kinetic energy of the ions, τ_c is the characteristic containment time in the heating volume, and P_{rf} is the electric power input into the heating antenna. In order to arrive at an approximation for this efficiency from first principles for an arbitrary RF heating process, we follow a Fokker-Planck formulation for the distribution function f_i of the ion ensemble:¹⁵

$$\frac{\partial f_i}{\partial t} + \nabla \cdot \mathbf{S} = 0 \quad (2)$$

where

$$\mathbf{S} = \mathbf{S}_w + \mathbf{S}_c, \quad (3)$$

where \mathbf{S}_w denotes the flux in velocity space due to the incident waves, \mathbf{S}_w and \mathbf{S}_c the flux due to interspecies collisions. The consecutive moments of this result yield the fluid equations. In particular, we are interested in the energy transport equation¹⁶

$$\frac{3}{2} \frac{dn_i T_i}{dt} + \frac{5}{2} n_i T_i \nabla \cdot \mathbf{u}_i = -\nabla \cdot \mathbf{Q}_i + \mathbf{R}_{i-s} \cdot \mathbf{u}_i - \left(\frac{\partial W}{\partial t} \right)_{i-s} + W_w \quad (4)$$

where we have assumed the plasma to be in local thermal equilibrium and isotropic with definable T_i and pressure. Q_i denotes the heat flux term, R_{i-s} is the drag on ions produced by other species in the plasma, u_i is the fluid velocity of the ions, W_w is the change in energy density with time introduced by the wave, and $(\frac{\partial W}{\partial t})_{i-s}$ is the energy source and sink terms due to inelastic collisions with the electrons and neutrals in the plasma. In order to simplify this expression further and relate it to the overall input power, we integrate Eq. 4 over the heating volume:

$$\int_V \left(\frac{\partial W}{\partial t} \right)_i dV = \int_V W_w dV. \quad (5)$$

Here we have assumed the plasma has a constant, steady-steady, internal energy, lumped all of the losses terms into a single term $(\frac{\partial W}{\partial t})_l$, on the left hand side, and made the reasonable assumption that the only source of power is the waves. In order to relate this expression to the canonical definition for efficiency, we approximate the lossy term as the increase in total energy over the background divided by the confinement time:

$$\int_V \left(\frac{\partial W}{\partial t} \right)_l dV = \frac{n_i V \Delta T_i}{\tau_c}. \quad (6)$$

This simplification allows us to express the canonical definition of efficiency in terms of the input wave power,

$$\eta = \frac{\int_V W_w dV}{P_{rf}}. \quad (7)$$

In this way, we have recast the efficiency to be in terms of the absorption of power from the wave, which will depend on a number of plasma and wave parameters. The goal of the next section is to find an analytical expression for these dependencies in a BEWH process.

III. Wave Power

A. Geometry

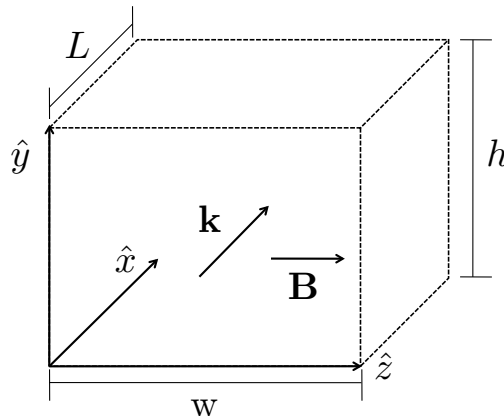


Figure 1. The simplified, rectilinear heating geometry. The uniform, magnetic field is in the \hat{z} direction. The waves propagate perpendicularly to this magnetic field in the \hat{x} direction with wavevectors parallel to k .

For the case of perpendicularly-propagating electrostatic waves, as is found in BEWH, we consider the simplified geometry shown in Fig. 1. In this configuration, the electrostatic waves are launched from the edge of the heating volume ($x = 0$) where they propagate in the \hat{x} direction that is perpendicular to the magnetic field ($\mathbf{B} = B_0 \hat{z}$). The incident area where the antenna excites the BEW modes is given by $A_w = h \times w$ while the heating volume is denoted V . The wave is assumed to be uniform in the plane orthogonal to the direction of propagation and to vanish outside the sample volume.

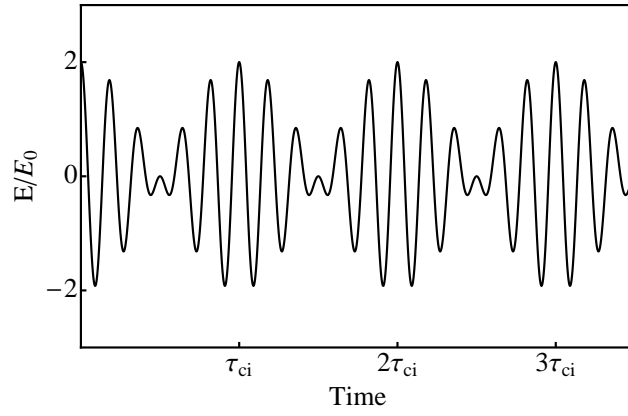


Figure 2. A sample wave pattern for BEW at a fixed point in space. The period of oscillation of the beat is given by $\tau_{ci} = 2\pi/\omega_{ci}$

The electric field for BEWH, which exhibits the beat pattern at the cyclotron frequency discussed in the previous section, is shown in Fig. 2 at a fixed position in space. This field is characterized as

$$\mathbf{E} = E_0(x) [\cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t)] \hat{x} \quad (8)$$

where $E_0(x)$ is the electric field amplitude that is a function of position in the plasma and k_1, k_2 denote the wavenumbers. For BEWH, we require that $\omega_2 = \omega_1 + \omega_{ci}$ and that the amplitudes of the waves are equal. These conditions have been shown⁵ to permit the waves to couple with the greatest range of ions with velocities outside the resonant condition ($v_{\perp} \sim v_{\phi}$). The phase difference between the waves has been neglected since this has a minimal impact on the BEW process.^{5,6}

In examining the wave propagation through the plasma for our efficiency calculations, we note that for very large wave amplitudes, self-consistent effects can significantly change the dispersion relation of the propagating waves.¹⁷ We avoid this outcome in our analysis, however, by assuming that the wave amplitudes are sufficiently small such that this effect never occurs. We can express this assumption explicitly in terms of timescales. In particular, the timescale τ_h for ion energy exchange with the waves and therefore the characteristic time for the saturation effect to occur has been shown to decrease with increasing wave amplitude.⁹ For sufficiently small wave amplitudes then, the saturation time will be long compared to the characteristic confinement time, $\tau_c \ll \tau_h$. Under these conditions, wave saturation will never occur since physically the BEW have only a very short time to introduce energy to an ion ensemble before the ions exit the heating volume.

Since the ion ensemble only has a limited interaction with the plasma, this restriction on timescales further allows us to make the simplifying assumption that the BEW will always act on a background plasma with initial temperature T_{i0} that enters the plasma heating volume. This assumption about plasma temperature relates closely to the test particle treatment described in detail by Karney¹³ and also utilized by Fisch to successfully model current drive in fusion plasmas.¹⁸ As we will discuss shortly, this assumption also allows the derivation of a simplified expression for wave heating.

Since the wave dispersion relation remains unchanged as the wave propagates through the plasma, we can express the flow of wave energy density through the plasma as¹⁹

$$\mathbf{v}_g \frac{\partial U}{\partial x} = -W_w, \quad (9)$$

where U denotes the energy density of the BEW and \mathbf{v}_g is the group velocity of the wave packet. Integrating over the volume then yields

$$A_w \mathbf{v}_g (U(L) - U(0)) = - \int_V W_w dV. \quad (10)$$

Finally, we note that at the incident surface of the heating volume, $A_w \mathbf{v}_g U(0) = \gamma P_{rf}$, where $\gamma \leq 1$ represents the coupling coefficient between the antenna and the excited mode in the plasma. Substituting into Eq. 7

thus yields

$$\eta = \gamma \left(1 - \frac{U(L)}{U(0)} \right). \quad (11)$$

B. Diffusion Coefficient

The challenge now is to solve for $U(x)$ in Eq. 9. We proceed by first modeling the flux term $\nabla \cdot \mathbf{S}_w$ with a diffusion operator. This prescription is valid provided the ion motion is stochastic, i.e. the motion becomes sufficiently uncorrelated to justify approximating the evolution of ions in phase space with a diffusion operator. While collisions can in part contribute to this stochasticity, it has been shown that both SEW and BEW can generate chaotic motion in particle trajectories by virtue of the nonlinear interaction of ions with the propagating waves. The range of perpendicular ion velocities v_\perp where this motion occurs depends on the amplitude of the wave. For the SEW case, Karney²⁰ demonstrated that ions with perpendicular velocities in the range $v_\phi - v_{tr} < v_\perp < G(E_0, v_\phi)$ —where $v_{tr} = \sqrt{eE_0/m_i k}$ is the trapping velocity and the upperbound $G(E_0, v_\phi)$ is a monotonically increasing function of both wave amplitude and phase velocity—will have orbits that evolve chaotically. Ions with velocities outside of this range have integrable motion (no diffusion). On the other hand, for BEW it has been shown by Spektor and Choueiri²¹ that the nonlinear interaction of the beats with the cyclotron precession can permit ions with velocities below the range for SEW to be first coherently and then chaotically accelerated by the waves. We model this effect by extending the range of uncorrelated motion for BEW to a lower bound such that $(v_\phi - v_{tr})/2 < v_\perp < G(E_0, v_\phi)$ we approximate $v_\phi = \omega_1/k_1$ as the phase velocity of the lower frequency wave and let $k = k_1$. The use of a diffusion term for the wave interaction is justified for this range of ion velocities.

With this in mind, we follow the treatment outlined by Karney¹³ to approximate the wave flux term with the following diffusion expression:

$$\nabla \cdot \mathbf{S}_w = \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp D(v_\perp) \frac{\partial}{\partial v_\perp} F(v_\perp), \quad (12)$$

where $F(v_\perp)$ denotes the distribution function in the perpendicular-velocity direction and D is the diffusion coefficient. Instead of deriving a diffusion coefficient independently for BEW from first principles, we remark that it has been observed numerically and theoretically^{3,5} that the diffusion of ions undergoing stochastic motion under SEW acceleration is similar to that in the BEW case. For BEWH, we therefore use a modified form of the diffusion coefficient derived from a combination of numerical and analytical considerations by Karney for SEW:¹³

$$D(v_\perp) = \begin{cases} \frac{\pi}{4} \frac{(E_0/B_0)^2 v_\phi^2 \omega_{ci}}{v_\perp^2} \left| H_\nu^{(1)}(\rho) \right|^2 g^2(A) & \text{if } v_\perp \geq v_\phi; \\ D(v_\phi) & \text{if } (v_\phi - v_{tr})/2 \leq v_\perp < v_\phi \\ 0 & \text{if } v_\perp < (v_\phi - v_{tr})/2 \end{cases}$$

where $\rho = kv_\perp/\omega_{ci}$; $\nu = \omega/\omega_{ci}$; $A = (\varepsilon\nu/\rho) \left| H_\nu^{(1)'}(\rho) \right|$; $\varepsilon = E_0 k/B_0 \omega_{ci}$; $H_\nu^{(1)}$ is the first derivative of the Hankel function of the first kind; and $g(A) = \max(1 - A_s^2/A^2, 0)$ with $A_s = 1/4$.

The diffusion coefficient is defined piecewise to capture the relative degree of stochasticity that occurs for ion orbits depending on their perpendicular velocities. The coefficient is maximized where $v_\perp = v_\phi$ as the ion experiences the most resonant interaction with the wave at this condition. As the ion phase velocity moves above this resonance, the diffusion coefficient naturally decreases. For $v_\perp < v_\phi$, Karney originally reported that the diffusion constant should assume the value $D(v_\phi)$ for $v_\phi - v_{tr} < v_\perp < v_\phi$ to reflect ion trapping that can occur in the wave potential. We, however, have extended this definition to $v_\perp = (v_\phi - v_{tr})/2$ for BEW to account approximately for the non-resonant acceleration produced by the beats. For velocities lower than this bound, the motion is roughly integrable such that the diffusion coefficient is approximately 0.

Armed with this expression, we can evaluate the power density introduced by the wave. From the definition of W_w in Eq. 4, we get

$$W_w = 2\pi n_i m_i \int v_\perp^2 D \frac{\partial F}{\partial v_\perp} dv_\perp. \quad (13)$$

C. Asymptotic Evaluation for Wave Power

In order to estimate W_w , we would need to integrate over the entire distribution and the piecewise defined D. However, this is prohibitively complicated and will not permit an analytical evaluation. In order to search for a more tractable solution, we make a series of simplifying assumptions that allow us to more easily evaluate Eq. 13. First, we assume that $v_{ti} \ll v_\phi$ where $v_{ti} = \sqrt{2T_{i0}/m_i}$ denotes the thermal velocity of the ions. This is applicable to a number of kinetic electrostatic modes.¹⁹ Second, we assume the wave amplitude to be sufficiently small such that $v_{tr} \ll v_\phi$. Finally, we employ the approximation cited above that the steady-state distribution is Maxwellian with a background temperature T_{i0} during the heating process:

$$F(v_\perp) = \left(\frac{m_i}{T_{i0}}\right) e^{-\frac{m_i v_\perp^2}{2T_{i0}}}. \quad (14)$$

With these assumptions, we can simplify Eq. 13 by retaining only the dominant term from the integration:

$$W_w = n_i m_i \frac{\pi (E_0/B_0)^2 \omega_{ci} m_i v_\phi^2}{4 (\omega/\omega_{ci})^{2/3} 4T_{i0}} e^{-\frac{m_i v_\phi^2}{2T_{i0}}}. \quad (15)$$

Rewriting this, we find a reduced expression:

$$W_w = \frac{n_i \pi m_i^2 (E_0/B_0)^2 \omega_{ci}^{5/3}}{16} \left(\frac{v_\phi^2}{kT_{i0}^{3/2}}\right)^{2/3} e^{-\frac{m_i v_\phi^2}{8T_{i0}}}. \quad (16)$$

This result represents the rate at which the energy density is increased at a point in a plasma due to interaction with the BEW. From this expression, we can gain several physical insights into the BEW heating efficiency:

- **BEWH superior to SEWH** Comparing Eq. 16 to the power term derived by Karney for SEW under similar assumptions regarding the heated plasma distribution (Eq. 101 in Ref. 13), we conclude that is larger than SEWH . This is to be expected as the non-resonant nature of BEW ion acceleration enables it to heat a larger fraction of a Maxwellian ion distribution.
- **Amplitude dependence** We see the expected result that the power deposition is a quadratic function of the wave amplitude. This has been observed experimentally in SEWH¹⁷ as well as in comparable non-collisional heating processes (Ref. 19, Chapt. 11), and it is a reasonable expectation for BEWH.
- **Impact of wavenumber** With longer wavelengths (small k), the rate of heating increases. This reflects the fact that as the electric field becomes increasingly uniform from the perspective of the particle ($|r_L|k < 1$ where r_L is the ion cyclotron radius), the electrostatic perturbation experienced by individual particles according to Eq. 8 approaches $E \sim \cos(\omega_{ci}t)$. This oscillation at the ions' fundamental frequency can lead to secular growth and extremely large increases in energy.
- **Role of phase velocity** For $v_{ti} \ll v_\phi$, the increase of P with the ratio of phase velocity to ion thermal velocity illustrates how an increasingly larger fraction of particles will be subject to interaction with the wave as the average velocity of the plasma approaches the BEW lower bound at $v_\phi/2$.

It is reassuring to note that the same physically intuitive trends were observed through the second order expression for the equilibrated kinetic energy of a collisionless ion ensemble subject to BEW, which we derived in Ref. 3. While the term in parenthesis in Eq. 16 is raised to the power 2/3, the corresponding term in the expression we derived in Ref. 3 appears to the first power. Also, the increase in heating we predicted at on-resonance conditions ($\omega_1 = n\omega_{ci}$ where n is an integer) does not figure explicitly into our above calculation for power from the BEWH. This effect could be incorporated with a corrective constant that is $\propto (\omega_1/\omega_{ci} - \|\omega_1/\omega_{ci}\| + \delta)^2$ where $\delta < 1$ and $\|\cdot\|$ denotes the nearest integer function in front of Eq. 16. However, for simplicity in the following investigation, we will assume this factor is $O(1)$ for all wave frequencies.

In its current form, Eq. 16 is applicable to electrostatic modes with phase velocities that are significantly higher than the ion thermal velocity. We should note that it loses validity when $(v_\phi/v_{ti})^2 < 4$ as the dominant term assumption employed to arrive at the above expression is violated. In this range of parameter space,

it is necessary to integrate Eq. 13 with the full definition of $D(v_{\perp})$. In spite of this limitation, Eq. 16 offers useful insight into the BEWH mechanism, and in the next section, we explore this result in more depth by examining its dependencies for a real electrostatic wave with a dispersion relation that constrains v_{ϕ} and k as functions of frequency.

IV. BEWH with EICW

In many laboratory and propulsion applications, the ionization fraction is relatively low and consequently it is necessary to chose a perpendicularly propagating electrostatic mode for BEW that will be relatively undamped by charge-exchange collisions. The Electrostatic Ion Cyclotron Wave (EICW) is an attractive mode for this purpose as it has been observed to not only persist in weakly ionized plasma but also to produce significant ion heating.^{10,11,22} In this section, we investigate Eq. 16 in conjunction with Eq. 9 for the case of ion cyclotron beating electrostatic wave heating, IC-BEWH and discuss the implications for both existing BEW experiments and future applications.

For a perpendicularly propagating EICW with small wavenumber in the magnetic field direction \hat{z} , the wave dispersion relation is given by¹⁹

$$k_x^2 \epsilon_{xx} + k_z^2 \epsilon_{zz} = 0, \quad (17)$$

where $\epsilon_{xx}, \epsilon_{zz}$ denote components of the dielectric tensor. For the frequency and energy conditions of the EICW, these components are given by

$$\begin{aligned} \epsilon_{xx} &= \frac{\omega_{pi}^2}{(\omega_{ci}^2 - \omega^2)^2}, \\ \epsilon_{zz} &= \frac{n_i e^2}{\epsilon_0 T_e k_z^2}, \end{aligned} \quad (18)$$

where ϵ_0 is the permittivity of free space and $\omega_{pi} = \sqrt{n_i e^2 / m_i \epsilon_0}$ is the plasma frequency. The approximate solution for Eq. 17 in the EICW domain is

$$\omega^2 = \omega_{ci}^2 + \frac{T_e}{m_i} k^2. \quad (19)$$

Now, assuming the wave to be largely perpendicularly propagating, $k = k_{\perp} \gg k_{\parallel}$, as is consistent with EICW and the BEW theory, the energy density of each mode is

$$U_j = \frac{\epsilon_0 E_0^2}{4} \frac{1}{2} \left(\frac{\partial \epsilon_{xx}}{\partial \omega} \right)_{\omega=\omega_j}, \quad (20)$$

where we have used the subscript $j = 1, 2$ to refer to the two waves. The energy flux for each wave is given by

$$\tilde{P}_j = \frac{\epsilon_0 \omega_j}{2k_j} (\epsilon_{xx})_{\omega=\omega_j}. \quad (21)$$

If we assume the two BEW not to exhibit any non-linear wave-wave interactions, then the total wave energy density is additive such that

$$U = \sum_{j=1,2} \frac{\omega_{pi}^2 \omega_j^2}{(\omega_{ci}^2 - \omega_j^2)^2} \frac{\epsilon_0 E_0^2}{2}. \quad (22)$$

The group velocity for power transfer of this combined packet of two waves is given by¹⁹

$$\mathbf{v}_g = \frac{\tilde{P}}{U} = \frac{\sum_{j=1,2} \frac{\omega_j}{2k_j} \frac{\omega_j^2}{(\omega_{ci}^2 - \omega_j^2)^2}}{\sum_{j=1,2} \frac{\omega_j^2}{(\omega_{ci}^2 - \omega_j^2)^2}}. \quad (23)$$

With this relationship and Eq. 16, we can solve Eq. 9 for two BEW with equal wave amplitude. This yields the following expression for wave energy density as a function of position in the plasma:

$$U(x) = U(0) e^{-\alpha x} \quad (24)$$

where the characteristic length scale is given by α^{-1} . This length constant in turn is

$$\alpha = \left(\sum_{j=1,2} \frac{\omega_j^2}{(\omega_{ci}^2 - \omega_j^2)^2} \right)^{-1} \frac{m_i}{\mathbf{v}_g \omega_{ci}^{1/3}} \frac{\pi}{8} \left(\frac{v_\phi^2}{k T_{i0}^{3/2}} \right)^{2/3} e^{-\frac{m_i v_\phi^2}{8 T_{i0}}}, \quad (25)$$

where as usual, $v_\phi = \omega_1/k_1$ and ω_1 is the smaller frequency of the two waves. This result yields a full expression for the energy density profile in the plasma for EICW. Since the dispersion relation of the waves is defined by Eq. 17, this result depends only on the initial ion temperature T_{i0} , the electron temperature T_e , and the frequency of the exciting waves where $\omega = \omega_1$, $\omega_2 = \omega + \omega_{ci}$. By combining this result with Eq. 11, we can finally estimate the efficiency of the IC-BEWH process to be

$$\eta = \gamma (1 - e^{-\alpha L}). \quad (26)$$

A. Comparison with BEW Experiments

The success of this heating efficiency model can be checked in part with results adapted from a previous experimental study—the Beating Waves Experiment II.¹¹ This second-generation experiment was constructed to examine the heating of ions due to two beating EICWs in a uniformly magnetized plasma. It consists of a cylindrical, uniformly magnetized argon plasma with a 16.5 cm diameter. The BEW in this heating experiment are excited by a loop antenna that launches waves perpendicularly to the magnetic field (similar to the geometry shown in Fig. 1). The height of the loop h from Fig. 1 is the same as the diameter of the cylinder, and the width of the heating volume w is 25.5 cm. The experimental parameters for this system are shown in Table 1 along with the estimated heating efficiencies according to Eq. 1. This calculated efficiency should not be confused with the so-called temperature efficiencies $\eta_h = (T_i - T_{i0})/T_{i0}$ reported for these experiments. For the calculation of $\eta(\text{Exp.})$, the heating containment time for this experiment was estimated as $\tau_e = w/v_{ti0}$, where v_{ti0} is the initial ion thermal velocity and w is the width of the heating zone in the direction of the magnetic field.

Table 1. Plasma parameters for BWX II reported in Ref. 11 along with the estimated and calculated efficiencies. T_i denotes the final ion temperature after heating

Parameter	BWX II
V	5440 cm ³
L	16.5 cm
n_i	10 ¹¹ cm ⁻³
T_e	4 eV
T_i	0.1 ± .01 eV
T_{i0}	0.05 ± .01 eV
P_{RF}	10 W
m_i	39.95 amu
ω_{ci}	157 kHz
ω	1.92 ω_{ci}
τ_e	7 ms
100 × η (Exp.)	0.006 ± .001%
100 × η (Cal.)	0.03%

The calculated efficiency according to Eq. 26 is also shown in this table. In this calculation, we have accounted for the fact that the heating in BWX II takes places in a cylindrical volume as opposed to the simplified rectilinear geometry used in the above derivations, by estimating the coupling coefficient γ as the ratio of the BWX II volume to a rectilinear sample volume with the same cross-section, A_w and length, L . This yields $\gamma = 0.78$.

It is evident from these considerations that the predicted efficiency according to the simple model and the measured efficiencies are the same order of magnitude, 0.006% vs. 0.03%. This agreement lends credence to our simplified one-dimensional analysis. The overestimation in efficiency should be expected given the ideal assumptions made in deriving Eq. 26. For example, BWX II does not have a uniform density and temperature, which could lead to reduced antenna coupling. Also, since the loop heating antenna is planar, it is very likely that significant heating power is radiated away from the plasma. This can change γ by another factor of 0.5 which would bring the calculated value closer to the experimental result.

The correspondence between experiment and model offers insight as to why the temperature increases $\Delta T_i < 0.1$ eV in BWX II have been so low. Indeed, the characteristic length of absorption α^{-1} for these plasma and wave parameters is approximately ~ 200 m. This suggests that the majority of the power of the wave passes directly through the plasma without coupling energy to it. The reason for this poor performance relates primarily to the high ratio of $(v_\phi/v_{ti})^2 \sim 50$. Indeed, as can be seen from the the exponential of Eq. 25, for a high ratio of velocities, the characteristic length scale for absorption becomes extremely long. This reflects the dependence of the BEW process on $v_\phi/2$ outlined in the previous section.

For applications such as electric propulsion that require a high efficiency heating stage, the extremely low values observed and predicted for BWX II efficiency are problematic. However, the plasma parameters in the laboratory experiment, which was designed to study fundamental aspects of the heating process, do not reflect the conditions under which BEWH would be employed for an electric propulsion concept. In the next section, we perform a simple analysis for a plasma characteristic of a thruster designed to utilize RF heating.

V. Extension to Electric Propulsion

In order to assess the efficiency of BEWH for an electrothermal propulsion concept, we show in Table 2 a series of plasma parameters typical for a high power electrothermal propulsion concept.^{23,24}

Table 2. Comparable parameters to VASIMR as reported in Ref. 23,24

Parameter	BWX II
L	9 cm
m_i	2 amu
T_e	10 eV
T_{i0}	0.7 eV
ω_{ci}	16 MHz

For this configuration, a fast moving plasma flow along an approximately uniform magnetic field passes through a finite heating stage. In order to adapt our BEW efficiency model to this configuration, we assume that a loop antenna is placed along the cross-section that is transverse to the magnetized plasma and that the antenna geometry is configured in such a way to achieve a coupling coefficient of $\gamma \approx 1$. The distance L corresponds to the diameter of the heated plasma. We note here that since $T_e/m_i \sim v_\phi^2 \gg v_{ti}^2$, our assumptions leading to the derivation of Eq. 26 remain approximately valid. Similarly, we see that since this configuration is characterized by a fast flow through the heating region, the test-particle assumption that the ensemble acted upon the wave is Maxwellian with T_{i0} is also approximately valid. Given these constraints, we plot in Fig. 3 the heating efficiency as a function of ω_1/ω_{ci} according to Eq. 26 with the parameters of Table 2.

We immediately see from this figure that with a judicious choice of wave frequency, our simple efficiency model predicts total coupling to the plasma. This stems from the fact that in this case, for $\omega/\omega_{ci} > 1.1$, the velocity ratio is $(v_\phi/v_{ti})^2 < 13$ – significantly lower than in the laboratory plasma. This improvement in efficiency of course must be tempered by the assumptions of ideal coupling of the RF into wave energy

($\gamma = 1$) as well as the one-dimensionality of the model; however, as a first order approximation, the BEWH process seems capable of producing significant heating efficiencies.

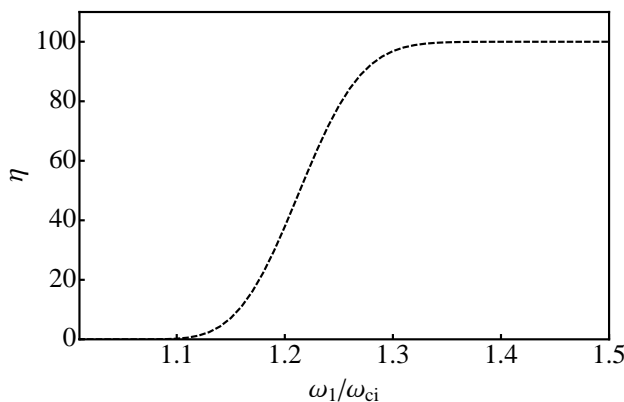


Figure 3. The predicted efficiency for plasma heating using BEWH in an electrothermal propulsion concept as a function of the ratio of wave frequency to ion cyclotron frequency.

We conclude this section with a note about the dependence of efficiency on the wave frequency exhibited in Fig. 3. The reason the efficiency drops to 0 below the ion cyclotron frequency stems from the cutoff of the EICW at $\omega = \omega_{ci}$ as can be seen from the dispersion relation in Eq. 19. On the other hand, we see that as ω_1 increases, the efficiency reaches a plateau. Physically, this corresponds to the wave being completely damped at some point $x < L$ before exiting the plasma. This suggests that the power deposition profile as determined by $U(x)$ may in fact also be controlled by the frequency of the waves ω_1, ω_2 . This is a useful consideration if uniform heating of the plasma is desirable.

VI. Discussion

From the derivations above, we see that the efficiency model for BEWH has provided an explanation for the low efficiencies observed in laboratory experiments and offers a means for uncovering a parameter space for achieving higher plasma coupling. There are of course limitations for this model, which have been outlined in brief. In particular, the level of heating that can be achieved is inherently limited not only by the simplicity of the model for power deposition but also by the assumption that the wave maintains the same dispersion relation. Since self-consistent effects can prevent the wave from propagating when $\tau_h \sim \tau_c$, the efficiency model therefore is only applicable below a maximal value in electric field where $\tau_c \ll \tau_h$. The validity of our simplified result in Eq. 16 is also limited by our assumption that $v_\phi \gg v_{ti}$.

Based on our expression in Eq. 26, it is reasonable to anticipate that as $v_{ti} \rightarrow v_\phi$, the power deposition and subsequent heating efficiency will also improve. However, in order to confirm this in the range $v_\phi \sim v_{ti}$, it would be necessary to perform a more detailed numerical calculation of W_w based on the full form of the diffusion coefficient given by D. Without going to this detail, we still can infer from the efficiency model that given the correct parameter space, BEWH should be able to produce significant heating at high efficiencies. The above results also suggest that the power deposition profile can also be modified by changing the wave frequency of excitation. Both of these features hold particular promise for plasma heating in an electrothermal thruster.

VII. Conclusion

In summary, we have used a Fokker-Planck analysis to derive an estimate for the heating of a plasma using BEWH. We have showed that this model can predict the coupling efficiency observed in laboratory experiments performed to date, and we have used our expression to demonstrate that for an electrothermal plasma propulsion concept, the BEW coupling may be extremely efficient. The successful implementation of this heating process depends on utilizing electrostatic ion cyclotron waves in an appropriately conditioned plasma. However, given these correct plasma parameters, we can see that BEWH represents a potentially important mechanism as a heating stage for both electrothermal propulsion and other plasma heating ap-

plications.

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