Implementation of externally-applied magnetic fields to a non-ideal MHD solver for MPD thruster simulation

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The Princeton Code for Advanced Plasma Propulsion Simulation (PCAPPS) is an axisymmetric MHD code for simulation of magnetoplasmadynamic thruster (MPDT) flow fields. This paper presents the derivation of a solver which allows for implementation of applied magnetic fields. In general, an applied magnetic field is necessary for MPDTs to operate efficiently at lower power levels. Several candidate methods are considered: use of a flux function, projection schemes, constrained transport methods, and 8-wave methods. The relative merits of these methods are assessed with emphasis on compatibility with non-ideal MHD terms and applicability to applied-field MPDT flows. The 'corrected' 8-wave method is selected because it satisfies the conservation criterion, maintains a divergenceless magnetic field, and is easily incorporated into simple and higher-order finite volume methods. The derivation of a first-order accurate, explicit 8-wave scheme for the axisymmetric, applied-field MHD equations is subsequently presented.

Nomenclature

$\mathbf{B}$	= magnetic inductance
$\mathcal{E}$	= energy density
$\mathbf{E}$	= electric field
$\bar{\bar{E}}_{res}$	= resistivity tensor
g	= torroidal flux function
$\mathbf{j}$	= current density
$\bar{\bar{k}}_{th}$	= thermal conductivity tensor
μ_0	= permeability of free space
n	= number density
p	= isotropic pressure
ψ	= poloidal flux function
$\mathbf{q}_{diss}$	= dissipative energy flux
ρ	= mass density
$\mathbf{v}$	= fluid velocity

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I. Introduction

The Princeton Code for Advanced Plasma Propulsion Simulation (PCAPPS) was originally developed to explore the physical processes of axisymmetric, self-field magnetoplasmadynamic thrusters (MPDTs).[1] The purpose of developing a code specifically for MPDT simulation was to allow for greater flexibility in implementing a variety of non-ideal physics, including resistivity with Hall effect, thermal conduction, anomalous transport, thermal non-equilibrium between ions and electrons, and a calorically imperfect equation of state. The code was benchmarked against experimental data for the Villani-H thruster and Full Scale Benchmark thruster, and used to study the physics of Lithium Lorentz Force Accelerators (Li-LFAs).

The original code was, however, limited in application to thrusters where the magnetic field is generated entirely by the plasma current. For lower-power MPDTs it is necessary to add an externally-applied magnetic field in order to achieve similarly high efficiency. This modification is not trivial from a numerical standpoint, since the addition of an applied magnetic field requires us to solve governing equations for B_r, B_z, and azimuthal momentum. It also introduces the possibility of non-zero divergence of the magnetic field.

Thus, the goal of this paper is to develop a numerical scheme for simulation of axisymmetric, applied-field MPDT flows. We begin in section 2 by outlining the governing equations for resistive MHD, with particular emphasis on the significance of having an applied magnetic field. Section 3 discusses several candidate methods for solving the system of equations, namely flux function formulations, projection schemes, constrained transport methods, and 8-wave methods. Having selected the corrected form of the 8-wave method, section 4 proceeds with the mathematical manipulations necessary to write the axisymmetric, mixed hyperbolic/parabolic system in two-dimensional cartesian form. Section 5 outlines the simple scalar diffusion numerical scheme which has been used in work thus far, and gives reference to methods which might be used to achieve higher-order accuracy.

II. Governing Equations

This paper presents a numerical scheme for the resistive MHD equations, a set of mixed hyperbolic and parabolic equations, which can be written in the following conservative form,

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \mathbf{B}\\ \mathcal{E}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{vv}+(p+\frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0})\bar{\bar{I}}-\frac{1}{\mu_0}\mathbf{BB}\\ \mathbf{vB}-\mathbf{Bv}\\ (\mathcal{E}+p+\frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0})\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{B}}{\mu_0}\mathbf{B}\end{bmatrix}=\nabla\cdot\begin{bmatrix}0\\ 0\\ \bar{\bar{E}}_{res}\\ \mathbf{q}_{diss}\end{bmatrix}. \tag{1}$$

The energy density is the sum of internal, kinetic, and magnetic field energies:

$$\mathcal{E}=\mathcal{E}_{int}+\rho\frac{\mathbf{v}\cdot\mathbf{v}}{2}+\frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}. \tag{2}$$

Closure of the system is achieved by imposing an equation of state which relates the pressure and temperature to the density and internal energy, $\mathcal{E}_{int}$. For Faraday's law, the tensor $\bar{\bar{E}}_{res}$ is defined such that

$$\nabla\cdot\bar{\bar{E}}_{res}=-\nabla\times(\mathbf{E}'), \tag{3}$$

where $\mathbf{E}'=\eta\mathbf{j}+(\mathbf{j}\times\mathbf{B}-\nabla p_e)/ne$ is an expression of Ohms' law for the electric field in the plasma reference frame.[2] This quantity also appears in the parabolic term of the energy equation, $\nabla\cdot\mathbf{q}_{diss}$, along with thermal conduction:

$$\mathbf{q}_{diss}=\frac{\mathbf{B}\times\mathbf{E}'}{\mu_0}+\bar{\bar{k}}_{th}\cdot\nabla T. \tag{4}$$

For the mathematical manipulation and discussion that follows, it is convenient to write the system of equations in the compact form

$$\frac{\partial\mathbf{U}}{\partial t}+\nabla\cdot\bar{\bar{\mathcal{F}}}_h=\nabla\cdot\bar{\bar{\mathcal{F}}}_p, \tag{5}$$

where $\mathbf{U}$ is the vector of conserved variables, $\bar{\bar{\mathcal{F}}}_h$ is the tensor of hyperbolic fluxes corresponding to convective physics, and $\bar{\bar{\mathcal{F}}}_p$ is the tensor of parabolic fluxes corresponding to dissipative physics.

A. Significance of Applied Magnetic Fields

For most MPDTs it is desirable to work in axisymmetric cylindrical coordinates - given an axisymmetric thruster geometry, it is reasonable to assume axisymmetry of the plasma flow. This assumption is also perhaps necessary to make direct simulation computationally feasible. Let us define a cylindrical coordinate system (r, θ, z) and assign to these coordinates the velocity components (u, v, w) and magnetic field components (B_r, B_θ, B_z) respectively.

For self-field MPDTs, in which the magnetic field is due entirely to a current density having only r and z components, it is easy to show (from Ampere's law) that the magnetic field is purely azimuthal. It follows that the the Lorentz force contribution to the momentum equation is also planar, and therefore the equation for azimuthal momentum is trivial. Eq. (1) then reduces to a system of five scalar equations governing the evolution of $\{\rho, \rho u, \rho w, B_\theta, \mathcal{E}\}$. It is worth noting that a pure azimuthal magnetic field is automatically divergenceless,

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \tag{6}$$

since B_r, B_z, and $\frac{\partial}{\partial \theta}$ are zero.

Introduction of an applied magnetic field significantly complicates the situation. Although this field is axisymmetric for MPDTs, its interaction with the plasma current reintroduces azimuthal momentum. For this case Eq. (1) can be written as a system of eight scalar equations governing the evolution of $\{\rho, \rho u, \rho v, \rho w, B_r, B_t, B_z, \mathcal{E}\}$. The system no longer guarantees that $\nabla \cdot \mathbf{B} = 0$, as least numerically, as we shall expand upon in the next section.

III. Numerical Methods for the MHD Equations

In this section we briefly survey numerical methods which have been used to solve two and three dimensional plasma flows. In doing so, we aim to identify a preferable numerical method for solving Eq. (1) with applied magnetic fields. Aside from the preferable qualities of low computational cost and memory space, and the requisite stability and convergence, an appropriate numerical method should

- Conserve mass, momentum, and energy
- Maintain $\nabla \cdot \mathbf{B} = 0$ as accurately as possible
- Have minimal numerical dissipation
- Enforce positivity of the density and energy

The conservation condition represents a strong physical law, and is probably the most important since a great deal of analysis is based on determining how the conserved quantities (e.g. energy) are distributed in the system. Fortunately it is not difficult to satisfy, since any finite volume method which computes the fluxes at the cell interfaces is inherently conservative.[3]

Analytically $\nabla \cdot \mathbf{B} = 0$ is always true since magnetic monopoles are not found in nature. In general, numerical methods may fail to satisfy this condition, leading to non-physical solutions and possible numerical instability. Two approaches are often taken to ensure that $\nabla \cdot \mathbf{B}$ remains small: 1) Formulating the magnetic field equations such that the magnetic field must be divergenceless, 2) Setting $\nabla \cdot \mathbf{B} = 0$ as an initial condition and employing a numerical scheme which preserves this accuracy to truncation error. For axisymmetric problems, the first method is exemplified by the flux function formulation, whereby the magnetic field is represented in terms of two scalar functions $\Psi(r, z)$ and $g(r, z)$,[4]

$$\mathbf{B} = \nabla \theta \times \nabla \psi + g \nabla \theta. \tag{7}$$

Since the divergence of the curl is zero, this formulation of the magnetic field is clearly divergenceless. One difficulty with this method is that the time-stepping equation for the poloidal flux funciton Ψ cannot be written in the conservative form used in Eq. (5). This prevents application of numerical schemes which calculate characteristics or linearize the flux tensor. The method is also of lower accuracy, since calculation of $\mathbf{B}$ requires taking a derivative.[5]

A number of methods fall into the second category for maintaining near divergenceless of the magnetic field. These include projection schemes, constrained transport methods, and the 8-wave formulation. The projection scheme was first applied to MHD by Brackbill and Barnes,[6] and essentially entails projecting the non-divergenceless solution for $\mathbf{B}$ onto the space of divergenceless solutions. Typically the projection step involves solving a Poisson equation. The constrained transport approach was introduced by Evans and Hawley,[7] and involves implementation of a finite difference discretization on a staggered grid which maintains $\nabla \cdot \mathbf{B}$ within truncation error. The 8-wave method suggested by Brackbill and Barnes[6] and employed by Powell[8,9] employs a non-conservative source term to the MHD system which allows numerically for the existence of magnetic monopoles. This modifies the eigensystem such that non-zero divergence of the magnetic field is convected out of the domain with the flow speed.

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \nabla \cdot (\mathbf{v}\nabla \cdot \mathbf{B}) = 0 \tag{8}$$

For a detailed comparison of numerical methods for maintaining $\nabla \cdot \mathbf{B} = 0$ see the article by Toth.[5]

One of the major shortcomings of Powell's 8-wave formulation is that it violates conservation of of momentum and energy. With Powell's correction, the ideal MHD equations can be written

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \mathbf{B} \\ \mathcal{E} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + (p + \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0})\bar{\bar{I}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \\ (\mathcal{E} + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0})\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{B}}{\mu_0}\mathbf{B} \end{bmatrix} = -\nabla \cdot B \begin{bmatrix} 0 \\ \mathbf{B} \\ \mathbf{v} \\ \mathbf{v}\cdot\mathbf{B} \end{bmatrix}, \tag{9}$$

which introduces a source wherever $\nabla \cdot \mathbf{B} \neq 0$. In numerical tests, Toth showed that this lack of conservation can produce incorrect jump conditions and shock speeds.[5] It was noted by Janhunen, however, the correction for magnetic divergence should only be applied to Faraday's equation.[10] This result was later derived by Dellar using relativist energy-momentum conservation taken to the non-relativistic limit.[11] Thus, the corrected 8-wave method preserves conservation of mass, momentum, and energy, while maintaining the property that non-zero divergence of $\mathbf{B}$ is convected out of the domain with the flow velocity.

For this paper we apply the corrected 8-wave method because it satisfies conservation and allows (since it is written as a divergence of a flux) for direct application of various finite-volume flux formulations (e.g. Rusanov, HLL, Roe) which can provide excellent shock-capturing (low numerical dissipation). Near divergenceless $\mathbf{B}$ should be satisfied for most thruster geometries since non-zero $\nabla \cdot \mathbf{B}$ is convected with the flow velocity, and regions of circulating or stagnant flow are atypical in MPDTs. Although the correction made by Janhunen and Dellar improves positivity, it is not guaranteed. However, we have observed that the occurrence of negative pressures in MPDT simulations is typically a transient phenomena occuring at the start of a simulation and confined to a small region where velocity gradients are high, or where the majority of the energy is magnetic. Numerical failure is often avoided in these instances by setting a lower bound on the density and pressure.

IV. Axisymmetric Integral-Form Equations

In this section we proceed by writing the MHD system (with the appropriate correction to Faraday's law) as a set of eight scalar equations, and evaluating the divergences in cylindrical coordinates. The resulting equations are integrated over a finite volume, resulting in an axisymmetric integral formulation to which the finite volume method can be applied.

A. Scalar Variable Representation

Using the previously discussed notation for cylindrical coordinates, the vector of conserved quantities is written as

$$\mathbf{U} = \begin{bmatrix} \rho & \rho u & \rho v & \rho w & B_r & B_\theta & B_z & \mathcal{E} \end{bmatrix}^T. \tag{10}$$

The hyperbolic flux tensor is similarly expanded into 8 scalar equations:

$$\bar{\bar{\mathcal{F}}}_h = \begin{bmatrix} \rho u & \rho v & \rho w \\ \rho u^2 + p + \frac{B^2/2 - B_r^2}{\mu_0} & \rho uv - \frac{B_r B_\theta}{\mu_0} & \rho uw - \frac{B_r B_z}{\mu_0} \\ \rho vu - \frac{B_\theta B_r}{\mu_0} & \rho v^2 + p + \frac{B^2/2 - B_\theta^2}{\mu_0} & \rho vw - \frac{B_\theta B_z}{\mu_0} \\ \rho wu - \frac{B_z B_r}{\mu_0} & \rho wv - \frac{B_z B_\theta}{\mu_0} & \rho w^2 + p + \frac{B^2/2 - B_z^2}{\mu_0} \\ 0 & uB_\theta - vB_r & uB_z - wB_r \\ vB_r - uB_\theta & 0 & vB_z - wB_\theta \\ wB_r - uB_z & wB_\theta - vB_z & 0 \\ \left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) u - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_r & \left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) v - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_\theta & \left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) w - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_z \end{bmatrix}. \tag{11}$$

In order to rewrite the divergence of the hyperbolic flux tensor in terms of r, θ and z derivatives, we employ the vector/tensor identities for divergence in cylindrical coordinates:[12]

$$\nabla \cdot \bar{\bar{\mathcal{F}}}_h = \frac{1}{r}\frac{\partial}{\partial r} \begin{bmatrix} r\rho u \\ r\left(\rho u^2 + p + \frac{B^2/2 - B_r^2}{\mu_0}\right) \\ r\left(\rho uv - \frac{B_r B_\theta}{\mu_0}\right) \\ r\left(\rho uw - \frac{B_r B_z}{\mu_0}\right) \\ 0 \\ r\left(uB_\theta - vB_r\right) \\ r\left(uB_z - wB_r\right) \\ r\left(\left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) u - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_r\right) \end{bmatrix} + \frac{1}{r}\frac{\partial}{\partial \theta} \begin{bmatrix} \rho v \\ \rho vu - \left(\frac{B_\theta B_r}{\mu_0}\right) \\ \rho v^2 + p + \left(\frac{B^2/2 - B_\theta^2}{\mu_0}\right) \\ \rho vw - \left(\frac{B_\theta B_z}{\mu_0}\right) \\ vB_r - uB_\theta \\ 0 \\ vB_z - wB_\theta \\ \left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) v - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_\theta \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} \rho w \\ \rho wu - \left(\frac{B_z B_r}{\mu_0}\right) \\ \rho wv - \left(\frac{B_z B_\theta}{\mu_0}\right) \\ \rho w^2 + p + \left(\frac{B^2/2 - B_z^2}{\mu_0}\right) \\ wB_r - uB_z \\ wB_\theta - vB_z \\ 0 \\ \left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right) w - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0} B_z \end{bmatrix} + \frac{1}{r} \begin{bmatrix} 0 \\ -\left(\rho v^2 + p + \frac{B^2/2 - B_\theta^2}{\mu_0}\right) \\ \rho vu - \frac{B_\theta B_r}{\mu_0} \\ 0 \\ 0 \\ vB_r - uB_\theta \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

The final non-derivative term, which arises from taking the divergence of a tensor in cylindrical coordinates, is referred to as the geometric source term.[3]

For the parabolic fluxes, we assume that $\mathbf{E}'$ may be calculated numerically.[1] Eq. (3) is then written in terms of derivatives in cylindrical coordinates,

$$\nabla \cdot \bar{\bar{E}}_{res} = -\begin{bmatrix} \frac{1}{r}\frac{\partial E_z'}{\partial \theta} - \frac{\partial E_\theta'}{\partial z} \\ \frac{\partial E_r'}{\partial z} - \frac{\partial E_z'}{\partial r} \\ \frac{1}{r}\frac{\partial (rE_\theta')}{\partial r} - \frac{1}{r}\frac{\partial E_r'}{\partial \theta} \end{bmatrix} = \frac{1}{r}\frac{\partial}{\partial r}\begin{bmatrix} 0 \\ rE_z' \\ -rE_\theta' \end{bmatrix} + \frac{1}{r}\frac{\partial}{\partial \theta}\begin{bmatrix} -E_z' \\ 0 \\ E_r' \end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix} E_\theta' \\ -E_r' \\ 0 \end{bmatrix} + \frac{1}{r}\begin{bmatrix} 0 \\ -E_z' \\ 0 \end{bmatrix}. \tag{13}$$

The parabolic component of the energy equation can also be expressed in this form,

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial (rq_r)}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}, \tag{14}$$

where the components of $\mathbf{q}$, as defined in Eq. (4), can be written

$$q_r = -\frac{E_\theta' B_z - E_z' B_\theta}{\mu_0} + k\frac{\partial T}{\partial r} \tag{15}$$

$$q_\theta = -\frac{E'_z B_r - E'_r B_z}{\mu_0} + k\frac{1}{r}\frac{\partial T}{\partial \theta} \tag{16}$$

$$q_z = -\frac{E'_r B_\theta - E'_\theta B_r}{\mu_0} + k\frac{\partial T}{\partial z}. \tag{17}$$

For completeness, we give the correction to Faraday's law here

$$\mathbf{S}_{\nabla\cdot B} = \left(\frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_r}{\partial z}\right)\left[\ 0\ 0\ 0\ 0\ u\ v\ w\ 0\ \right]^T. \tag{18}$$

B. Axisymmetric Finite Volume Integral

To apply the finite volume method, the differential conservation equations are integrated over an general finite volume V in cylindrical coordinates:

$$\int_V \left(\frac{\partial \mathbf{U}}{\partial t} + \nabla\cdot\bar{\bar{\mathcal{F}}}_h = \nabla\cdot\bar{\bar{\mathcal{F}}}_p + \mathbf{S}_{\nabla\cdot B}\right) r dr d\theta dz. \tag{19}$$

The assumption of axisymmetry causes the θ-derivative component to drop out of $\nabla\cdot\bar{\bar{\mathcal{F}}}_h$, $\nabla\cdot\bar{\bar{\mathcal{F}}}_p$, and $\mathbf{S}_{\nabla\cdot B}$, making integration over the interval $[\theta_1,\theta_2]$ simple. By interchanging the order of integration the θ-integral is evaluated, with the resulting constant canceling on either side of the equality. Applying the remaining integration to each term, and interchanging the order of differentiation/integration for the vector of conserved variables,

$$\frac{\partial}{\partial t}\int_A (r\mathbf{U})\, dr dz + \int_A \left(r\nabla\cdot\bar{\bar{\mathcal{F}}}_h\right) dr dz = \int_A \left(r\nabla\cdot\bar{\bar{\mathcal{F}}}_p\right) dr dz + \int_A (r\mathbf{S}_{\nabla\cdot B})\, dr dz, \tag{20}$$

where A is the cell area in the r,z plane. In the latter three terms, multiplying through by r eliminates the $1/r$ factor on the r-derivative and the geometric source vector. Carrying the r factor inside the z-derivative allows the terms to be expressed as cartesian divergences in (r,z). Multiplying the B_θ equation by $1/r$ eliminates the geometric source term. From here it is easy to see that the system of equations can be expressed equivalently in two-dimensional cartesian coordinates:

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}_h}{\partial r} + \frac{\partial \mathbf{G}_h}{\partial z} + \frac{\partial \mathbf{F}_p}{\partial r} + \frac{\partial \mathbf{G}_p}{\partial z} = \mathbf{S}, \tag{21}$$

where

$$\mathbf{Q} = \begin{bmatrix} r\rho \\ r\rho u \\ r\rho v \\ r\rho w \\ rB_r \\ B_\theta \\ rB_z \\ r\mathcal{E} \end{bmatrix} \quad \mathbf{F}_h = \begin{bmatrix} r\rho u \\ r\left(\rho u^2 + p + \frac{B^2/2 - B_r^2}{\mu_0}\right) \\ r\left(\rho uv - \frac{B_r B_\theta}{\mu_0}\right) \\ r\left(\rho uw - \frac{B_r B_z}{\mu_0}\right) \\ 0 \\ (uB_\theta - vB_r) \\ r\,(uB_z - wB_r) \\ r\left(\left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right)u - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0}B_r\right) \end{bmatrix} \quad \mathbf{G}_h = \begin{bmatrix} r\rho w \\ r\left(\rho wu - \left(\frac{B_z B_r}{\mu_0}\right)\right) \\ r\left(\rho wv - \left(\frac{B_z B_\theta}{\mu_0}\right)\right) \\ r\left(\rho w^2 + p + \left(\frac{B^2/2 - B_z^2}{\mu_0}\right)\right) \\ r\,(wB_r - uB_z) \\ (wB_\theta - vB_z) \\ 0 \\ r\left(\left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right)w - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0}B_z\right) \end{bmatrix}.$$

$$\mathbf{F}_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -E'_z \\ rE'_\theta \\ -r\,q_r \end{bmatrix} \quad \mathbf{G}_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -rE'_\theta \\ E'_r \\ 0 \\ -r\,q_z \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 \\ \rho v^2 + p + \frac{B^2/2 - B_\theta^2}{\mu_0} \\ -\left(\rho vu - \frac{B_\theta B_r}{\mu_0}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (\nabla\cdot\mathbf{B}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u \\ v \\ w \\ 0 \end{bmatrix} \tag{22}$$

V. Numerical Scheme

The system given in Eq. (21) can be solved using any of the standard conservative finite volume methods. Work on the PCAPPS code thus far has utilized a simple explicit method,[1] which does not require approximation of the Jacobian matrix or its eigenstructure. Shown here for one dimension, we discretize Eq. (21) as

$$\frac{\mathbf{Q}_j^{n+1} - \mathbf{Q}_j^n}{\Delta t} + \frac{(\mathbf{G}_h)_{j+1/2} - (\mathbf{G}_h)_{j-1/2}}{\Delta z} + \frac{(\mathbf{G}_p)_{j+1/2} - (\mathbf{G}_p)_{j-1/2}}{\Delta z} = S. \tag{23}$$

The flux at the boundary is approximated by

$$\mathbf{G}_{j+1/2} = \frac{1}{2}\left(\mathbf{G}_j + \mathbf{G}_{j+1}\right) - \mathbf{D}_{j+1/2}, \tag{24}$$

where $\mathbf{D}$ is numerical dissipation necessary to stabilize the method. The numerical dissipation can be formulated using higher-order features such as flux-limited anti-diffusion, however for this simple formulation we use scalar diffusion:

$$\mathbf{D}_{j+1/2} = \frac{1}{2}\left|\lambda_{max}\right|\left(\mathbf{Q}_{j+1} - \mathbf{Q}_j\right), \tag{25}$$

where $|\lambda_{max}|$ is the largest eigenvalue of the flux Jacobian.

Scalar diffusion schemes of this form, while computationally efficient, tend to artificially smooth out solutions. It should be straightforward to employ an explicit characteristic splitting method with flux limiters in order to achieve higher-order accuracy.[13] For exmple, in correcting Powell's 8-wave method, Janhunen developed an explicit HLL-Roe method for ideal MHD.[10] An implicit scheme was developed by Jones et al. for non-linear MHD using Powell's original 8-wave method.[14]

VI. Conclusion

We presented an axisymmetric, non-ideal MHD solver which is intended for simulation of magnetoplasmadynamic thrusters with an applied magnetic field. In comparing the candidate methods, it was found that the corrected form of the 8-wave method, as presented by Janhunen[10] and Dellar,[11] possessed several of the desired characteristics; it conserves mass, momentum and energy, maintains near-zero divergence of the magnetic field, and can easily be incorporated into higher-order finite volume methods. The mathematical manipulations necessary to apply the axisymmetric finite volume integral were provided, leading to the final form given in Eqs. (21) and (22). We concluded by providing the first-order accurate, scalar diffusion numerical scheme used in work thus far, and by suggesting that methods combining flux limiters and/or implicit time stepping could be implemented to enable higher-order accuracy.

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