Effects of Ion Collisions on Ion Acceleration by Beating Electrostatic Waves.

R. Spektor* and E.Y. Choueiri†
Electric Propulsion and Plasma Dynamics Laboratory (EPPDyL)
Mechanical and Aerospace Engineering Department
Princeton University, Princeton, New Jersey 08544

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A numerical model of the nonlinear interaction between beating electrostatic waves and magnetized ions, including collisions, is presented. Previous studies of the beating electrostatic waves (BEW) interacting with a single ion showed the ability of this mechanism to accelerate ions from arbitrarily low initial velocities, and have revealed the fundamental conditions for this interaction to occur. The present study extends the analysis to a large number of ions and includes ion-ion collisions. The numerical investigation combines a dynamical description for the ion-wave interaction and a Monte Carlo simulation of the collisions. Despite the thermalization role of collisions BEW acceleration was found to yield larger heating rate and higher particle energies than the better known interaction with the single electrostatic wave (SEW).

I. INTRODUCTION

Acceleration of magnetized ions by beating electrostatic waves (BEW) is a nonlinear phenomenon that may be occurring in nature and may have interesting applications to various problems including spacecraft propulsion. Observations made with the Topaz 3 rocket [1] indicated that ions are accelerated, in a region of natural electrostatic wave activity, in the topside ionosphere to the escape velocity. A puzzling issue is that initial velocities of these ions are significantly below the previously known threshold required for resonant acceleration by electrostatic waves. The threshold was derived in the context of ion interaction with a single electrostatic wave (SEW) [2, 3].

Benisti *et al*. [4, 5] proposed a non-resonant acceleration mechanism that relies on nonlinear interaction of an ion with a pair of beating waves. They showed that if the criterion

$$n\omega_c = \omega_2 - \omega_1, \tag{1}$$

is satisfied between any pair of electrostatic waves, ions can be accelerated from an arbitrary low initial velocity. Equation (1) states that the difference between the frequencies of the two beating waves ω_1 and ω_2 should equal to an integer multiple, n, of the ion cyclotron frequency ω_c. This is in great contrast with the well known SEW-ion interaction studied theoretically by Karney *et al*. [2, 3], Zaslavsky *et al*. [6, 7], and Chia *et al*. [8, 9], and experimentally by Skiff *et al*. [10]. These studies showed that for acceleration to take place ion initial velocity has to be within a resonance band of the wave velocity. Another fundamental understanding obtained from these studies was that the ion motion during SEW-ion acceleration is always stochastic.

Choueiri and Spektor [11] investigated the beating wave acceleration mechanism theoretically and found that while Eq. (1) is *necessary* for the acceleration to occur, it is not a sufficient condition. Spektor and Choueiri [12] derived and verified the necessary *and* sufficient conditions for interaction, also discussed in section II. When these conditions are satisfied, an ion with an arbitrary low initial velocity can accelerate through a regular (non-stochastic) motion in the electric field of the beating waves, then reach a threshold above which acceleration continues more vigorously (stochastically).

Since the BEW acceleration can increase the perpen-

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*Graduate Student, Research Assistant.
†Chief Scientist at EPPDyL.
Associate Professor, Princeton University, Applied Physics Group.
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dicular velocity of *all* ions, as opposed to only the resonant part of the distribution function, it is of particular interest to spacecraft propulsion applications where acceleration or heating efficiency is of prime importance. In order to obtain the first indication of the existence of this mechanism we have designed and built a dedicated experiment using a helicon source and RF antenna to launch pairs of beating waves [13].

In order to guide the design of the experiment and help in interpreting its results we needed a model of the interaction that more resembles the case of a real plasma than does the single ion model. Such a model should account for the interaction of the waves with large amount of particles, and most importantly ion-ion collisions. In this paper we present such a model based on using Monte Carlo techniques to describe collisions, and solving the equation of motion between collisions. We use the simulation to study parametrically the effects of ion collisions on the heating rate and attainable average energy for both SEW and BEW.

In section II we review the collisionless model that describes the interaction of a single particle with a spectrum of electrostatic waves. We also review previous findings resulting from that model. In section III we present the numerical model that allows tracking a large number of ions and account for a finite collision rate. In section IV we present and discuss the results of our numerical investigation, and in section V we summarize our findings and deduce a phenomenological picture that illustrates the fundamental differences between BEW and SEW ion acceleration.

II. SINGLE PARTICLE MODEL

A theoretical model for beating electrostatic waves interacting with a single ion is given in [4, 5, 12]. The description, which latter in this paper is augmented with inclusion of collisions and the ability of tracking many ions, is shown schematically in Fig. 2. The schematic shows an ion in a constant magnetic field B$\hat{z}$ and electrostatic wave traveling in transverse direction $\hat{x}$. The wave interacts with a gyrating ion causing a change in its Larmor radius. Because the magnetic field is constant, an increase in the Larmor radius directly corresponds to the increase in the ion's perpendicular velocity and thus its kinetic energy. The equation of motion governing the interaction between a spectrum of propagating electrostatic waves and a single ion can be easily derived [2, 8]:

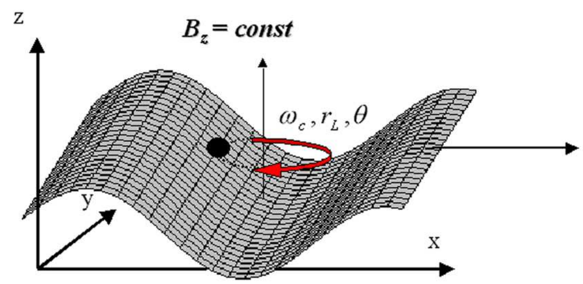

FIG. 1: A single ion of charge q and mass m in a constant homogeneous magnetic field B$\hat{z}$ interacts with an electrostatic wave. The wavenumber and electric field of the wave is parallel to the x-axis.

$$\frac{d^2x}{dt^2} + \omega_c^2 x = \frac{q}{m}\sum_i E_i \sin(\mathrm{k}_i x - \omega_i t + \varphi_i), \qquad (2)$$

where x and t are the coordinate and the time variables, q and m are the charge and the mass of the ion, $\omega_c = qB/m$, E_i, k_i, ω_i, and ϕ_i are the amplitude, wave number, frequency and phase of the i^{th} electrostatic wave. While Fig. 2 shows a single wave, a similar picture can be drawn for a spectrum of electrostatic waves traveling in the same direction. It is convenient to normalize the above equation and express it in the canonical form [14, 15]:

$$H = \rho^2/2 + \sum_i \frac{\varepsilon_i}{\kappa_i}\cos(\kappa_i \rho \sin\theta - \nu_i \tau + \varphi_i), \qquad (3)$$

where H is the Hamiltonian of the system, $\kappa_i = \mathrm{k}_i/\mathrm{k}_1$, $\nu_i = \omega_i/\omega_c$, $\tau = \omega_c t$, $\varepsilon_i = (\mathrm{k}_1 q E_i)/(m\omega_c^2)$, $\rho^2 = X^2 + \dot{X}^2$, and $X = \mathrm{k}_1 x$, $\dot{X} = dX/d\tau$, so that $X = \rho\sin\theta$, $\dot{X} = \rho\cos\theta$. Where θ is the cyclotron rotation angle measured clockwise from the y-axis as shown in Fig. 1 , and ρ is the normalized Larmor radius. Equations (2) and (3) could be solved numerically with either conventional 4^{th} order Runge-Kutta scheme or a symplectic approach. We have used the symplectic integration method developed by Candy and Rozmus [16] to study the behavior of a single ion interacting with one or two propagating electrostatic waves.

We were able to confirm [12] that while a single electrostatic wave produces some ion heating under restricted (resonance) conditions, two beating waves can result in ion acceleration from arbitrary low initial velocities. We have also shown that Eq. (1) describes the *necessary*, but not *sufficient* condition for that heating to take place. The necessary *and* sufficient conditions for ion heating by the beating electrostatic waves are [12]:

$$n\omega_c \; = \; \omega_2 - \omega_1, \qquad (4a)$$
$$H(\rho;\theta) \; = \; H_H > H(\rho \simeq \nu - \sqrt{\varepsilon};\theta = \pi), \quad (4b)$$

where H_H is the value of the Hamiltonian evaluated at the hyperbolic point as described in Ref.[12] and shown schematically in Fig. 2.

The schematic shows the possible acceleration processes. Particle 1 is accelerated stochastically in both cases. Particle 2 with initial energy below the SEW resonance threshold ($\rho = \nu - \sqrt{\varepsilon}$) is affected by BEW but is never allowed to reach the stochastic region. While particle 3 remains unaffected by the SEW interaction, it can be effectively accelerated by BEW through regular (non-stochastic) motion that allows it to reach stochastic region, where more rigorous acceleration takes place.

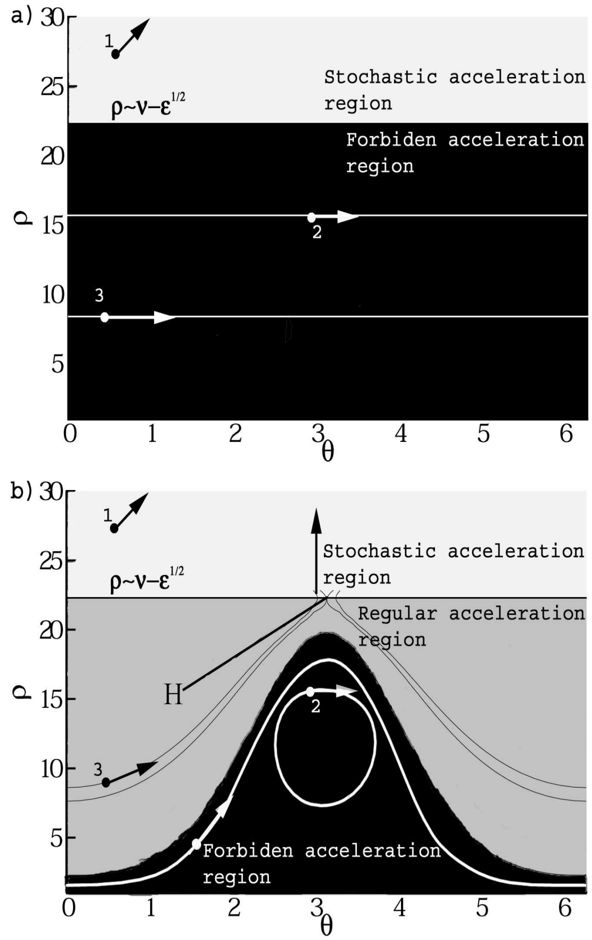

FIG. 2: Poincare cross-section (phase diagram) schematic. The figure shows typical trajectories for various initial conditions of ion interacting with: a) a single electrostatic wave (SEW), b) beating electrostatic waves (BEW).

III. INCLUDING COLLISIONS

Collisions alter the picture described in the previous section drastically. Without collisions an ion whose initial velocity (Hamiltonian) is below that corresponding to Eq. (4b), particle 2 in figure 2b will never be effectively accelerated by the waves. However, a collision would instantaneously change that ion's trajectory and place it in a part of phase place where Eq. (4b) is satisfied.

In this section we consider ion collisions only. Coulomb ion collisions are of interest since they thermalize the heavy species energy and since our main focus is ion heating.

To introduce collisions into our numerical model we follow the classical work of Takizuka and Abe [17]. We model Coulomb collisions as small angle binary collisions and assume that on a sufficiently small time scale we can uncouple particle motion from collisions. Thus our algorithm consists of two parts. We move all particles between collisions according to the equation of motion prescribed by the single particle collisionless model, Eq. (2). We then use the Monte Carlo approach to determine randomly the collision partners and the scattering angles for each collision.

A. Overall implementation

1. We first choose a time step Δt smaller than the ion-ion relaxation time calculated at the initial temperature of the ions.

2. Using a 4^{th} order Runge-Kutta scheme we then follow each particle in our simulation for Δt seconds according to the equation of motion for a single ion, Eq. (12).

3. Next we randomly choose a collision partner for each ion.

4. Using Monte Carlo method we then determine the

velocity increments for all colliding pairs as described in section III B. The new velocities are fed back into the Runge-Kutta solver.

5. After each collision, we store the value of the scattering angle Θ for each particle. We assume that whenever $\sum \sin^2\Theta \geq 1$ the particle has undergone one ion-ion Coulomb collision. Here the summation is over successive collisions for a given particle.

B. Momentum exchange during a collision

We treat Coulomb collisions between ions as a small angle binary elastic scattering events [17]. Such collisions preserve energy and momentum.

The relative velocity vector $\mathbf{u}(u_x, u_y, u_z)$ for a colliding pair is:

$$\mathbf{u} = \mathbf{v}_a - \mathbf{v}_b, \tag{5}$$

where $\mathbf{v}_a$ and $\mathbf{v}_b$ are the velocities of two colliding ions. The post-collision relative velocity $\mathbf{u}^f$ is:

$$\mathbf{u}^f = \mathbf{u}^i + \Delta\mathbf{u}, \tag{6}$$

where $\mathbf{u}^i$ is the relative velocity right before the collision and $\Delta\mathbf{u}$ is the change in the relative velocity due to the collision. The change in the relative velocity due to a scattering event could be derived from conservation principles [17],

$$\begin{aligned}\Delta u_x &= (u_x/u_\perp)u_z \sin\Theta\cos\Phi - (u_y/u_\perp)u\sin\Theta\sin\Phi \\ &- u_x(1-\cos\Theta), &\text{(7a)}\\ \Delta u_y &= (u_y/u_\perp)u_z \sin\Theta\cos\Phi + (u_x/u_\perp)u\sin\Theta\sin\Phi \\ &- u_y(1-\cos\Theta), &\text{(7b)}\\ \Delta u_z &= -u_\perp\sin\Theta\cos\Phi - u_x(1-\cos\Theta), &\text{(7c)}\end{aligned}$$

where $u_\perp^2 = u_x^2 + u_y^2$ and $u^2 = u_\perp^2 + u_z^2$. Here perpendicular and parallel directions are defined relative to the magnetic field ($\hat{z}$-axis). When $u_\perp = 0$ we have,

$$\begin{aligned}\Delta u_x &= u\sin\Theta\cos\Phi, &\text{(8a)}\\ \Delta u_y &= u\sin\Theta\sin\Phi, &\text{(8b)}\\ \Delta u_z &= -u(1-\cos\Theta). &\text{(8c)}\end{aligned}$$

Angle Φ is chosen homogeneously randomly from 0 to 2π. Angle Θ is chosen according to:

$$\sin\Theta = \frac{2\delta}{1+\delta^2}, \tag{9a}$$

$$1-\cos\Theta = \frac{2\delta^2}{1+\delta^2}, \tag{9b}$$

where $\delta = \tan(\Theta/2)$ is a random number chosen with the Gaussian distribution centered around zero and having the following variance $\langle\delta^2\rangle$:

$$\langle\delta^2\rangle = \Delta t \frac{q^4 n\lambda}{\pi\epsilon_0^2 m^2 \|\mathbf{u}\|^3}, \tag{10}$$

where q and m are the charge and the mass of the ion, n is the particle number density, λ is the Coulomb logarithm, ϵ_0 is the permittivity of free space, $\|\mathbf{u}\|$ is relative speed of two colliding ions, and Δt is the time step [17]. This small angle restriction allows us to interpret as the ion-ion collision frequency ν_{ii} to the binary collision frequency ν_b and the formalism implicitly accounts for electron shielding. We did not however track the electron dynamics as electrons are not expected to be effected by low frequency of the wave. We assume that the electron temperature is not affected by the low frequency waves of the problem, $\omega < \omega_{ce}$.

The post-collision velocity of each particle is found simply from,

$$\mathbf{v}_a^f = \mathbf{v}_a^i + (m/2)\Delta\mathbf{u}, \tag{11a}$$

$$\mathbf{v}_b^f = \mathbf{v}_b^i + (m/2)\Delta\mathbf{u}. \tag{11b}$$

C. Moving the particles

Starting from the Lorentz force equation,

$$\mathbf{F} = m\ddot{\mathbf{x}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{12}$$

we can derive equations of motion for a single particle in three dimensions. In our analysis, the magnetic field is constant ,$\mathbf{B} = \mathrm{B}\hat{z}$, and the electric field arises from the propagating electrostatic waves, as shown in Fig. 1,

$$\begin{aligned}\ddot{x} &= \dot{y} + E\sum_i \sin(x-\omega_i t), &\text{(13a)}\\ \ddot{y} &= -\dot{x}, &\text{(13b)}\\ \ddot{z} &= 0, &\text{(13c)}\end{aligned}$$

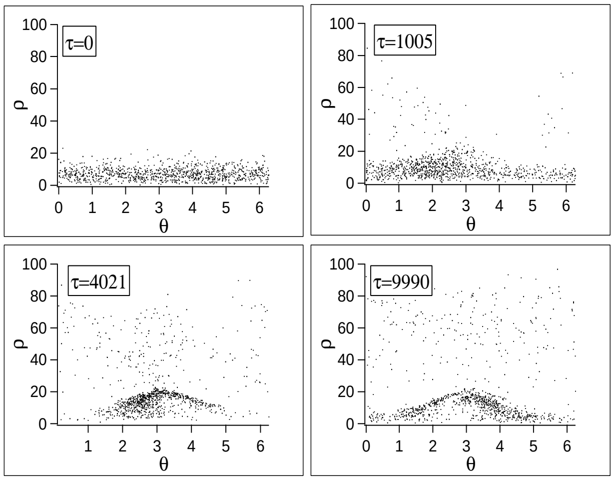

FIG. 3: $\varepsilon = 10,\ \nu_1 = 24.3,\ \nu_2 = 25.3$. Collisionless case. The "hump" corresponds to the particles not accelerated by the beating waves in accordance with Eq. (4).

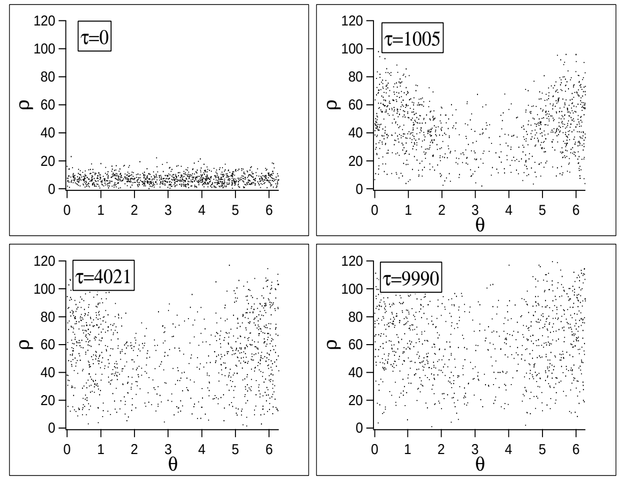

FIG. 4: $\varepsilon = 10,\ \nu_1 = 24.3,\ \nu_2 = 25.3$. Particles are allowed to collide with each other. We take the initial collision frequency required for step 1 of our algorithm to be $\sim 10^6$, which corresponds to $n_e \sim 10^{12}\ cm^{-3}$ and $T_e = 300$ K. Unlike Fig. 3, beating electrostatic waves accelerate all ions.

so that $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ are the second derivatives with respect to time t, and the other variables are the same as those appearing in equations (2) and (3). Equations (13) could be solved numerically using 4th order Runge-Kutta method.

IV. SIMULATION

The above model is used to simulated the case of BEW acceleration ($\varepsilon = 10,\ \nu_1 = 24.3,\ \nu_2 = 25.3\ \kappa_i = 1$) and compared to those of SEW acceleration under similar condition ($\varepsilon = 10,\ \nu = 24.3$). To visualize the numerical results we use Poincare cross-sections (ρ vs. θ phase diagrams) [15]. We will also investigate how collisions influence the energy evolution of the entire system.

A. Phase diagrams

Figure 3 follows collisionless evolution of 1000 particles in the plot ρ vs. θ. Initially we distribute all particles homogeneously over region of the phase space $\rho \lesssim 20$. The stochastic heating is observed whenever ions reach the *stochastic* zone ($\rho > 20$), Fig2b. Particles with initial conditions lying outside prohibited zone are accelerated as could be seen from that figure. The points corresponding to unaccelerated particles define a mount-like structure, seen in the last two panels of Fig. 3, which corresponds to prohibited zone shown in figure 2b. Figure 4 shows the BEW ion acceleration case where the evolution of phase space points is qualitatively different then the collisionless case illustrated in 3. Even ions originally in the forbidden acceleration zone are accelerated.

B. Energy Evolution

Now that we showed that collisions enhance ion heating, we will analyze the energy evolution of the entire system. In this section we compare the cases of BEW and SEW ion acceleration by beating electrostatic waves and a single electrostatic wave.

Figure 5 shows perpendicular component of the energy for BEW as well as SEW cases. Initially the energy increases exponentially. This corresponds to ions funnelling to the stochastic region. The process is analogous to phase space diffusion - thus its exponential nature. As more particles find their way into the stochastic region, the exponential increase is followed by the equilibration of the energy (by stochastic motion and collisions). Because particles fill up the stochastic region randomly, the statistical average of the energy stays constant.

Figure 5 demonstrates that collisions significantly increase both the ion heating rate and the final average energy.

In order to help us further illustrate these effects better

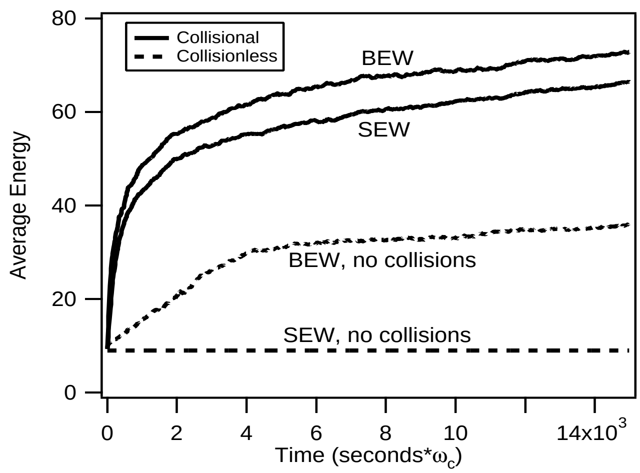

FIG. 5: Perpendicular energy evolution for 1000 particles interacting with beating waves. $\varepsilon = 10$, $\nu_1 = 24.3$, $\nu_2 = 25.3$.For comparison we also show the energy evolution for the single wave-ion interaction. $\varepsilon = 10$, $\nu = 24.3$.

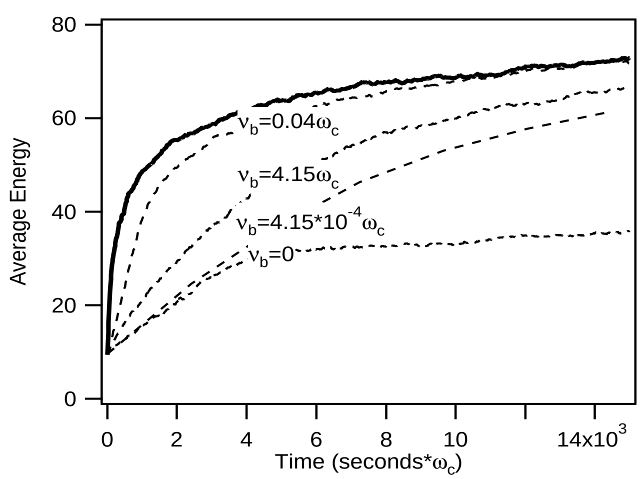

FIG. 6: Perpendicular energy evolution for 1000 particles interacting with beating waves. Dashed lines correspond to the clamped values of collision frequency and solid curve represents the self-consistent simulation. $\varepsilon = 10$, $\nu_1 = 24.3$, $\nu_2 = 25.3$.

we have ran the simulation with the collision frequency restrained to a constant value irrespective of the temperature. Figure 6 shows the results for four such cases and a simulation with self-consistent collision frequency calculation, for comparison. The figure indicates that there is an optimum collision frequency at which the heating rate and efficiency are at the maximum. When the collision frequency is increased from 0 to $\nu_b = 0.04\omega_c$, the ions diffuse faster into the stochastic region and the heating efficiency improves. However, as the collision frequency is further increased ($\nu_b = 4.14\omega_c$), the heating rate as well as the efficiency start dropping because collisions increasingly disturb ion motion too much. In a real plasma the ion collision frequency scales as $\sim T_{ion}^{-3/2}$ (for $T_i \lesssim T_e$). As the electrostatic waves deposit their energy into the plasma, the ions collide less often and the collision frequency drops to its optimum value. However, if the collision frequency decreases even further, the heating efficiency drops, driving down the temperature and stabilizing the collision frequency back to its optimum value. Therefore, we can conclude that the collision frequency always changes to accommodate the maximum possible heating rate. This becomes more evident by comparing the clamped value of ν_b simulations (dashed lines) to the solid line which was obtained by running the simulation and allowing the collision frequency to change self-consistently with the ion temperature evolution.

V. CONCLUSIONS

Numerical simulations of the nonlinear interaction of magnetized ions with beating electrostatic waves (BEW) were carried out. The resulting particle heating was compared to that obtained from simulations with interaction with SEW under the same conditions. The higher heating rate and temperature attained in BEW acceleration can be explained through the following fundamental description. In the SEW interaction thermalizing collisions are the only means for particles below a certain threshold of energy, associated with the resonant condition, to reach the region of phase space were stochastic and vigorous heating takes place. The non-resonant character of BEW acceleration allows a significant fraction of the ion distribution function to be accelerated and reach the stochastic region. This acceleration is augmented by collisions. In addition to the thermalizing role of collisions, this simulation-supported phenomenological picture points to the promise of using BEW as a new and efficient method for accelerating magnetized ions in a real plasma.

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