Electron Dynamics in a Beating Electrostatic Wave Magnetic Null Thruster

Matthew S. Feldman* and Edgar Y. Choueiri†

Electric Propulsion and Plasma Dynamics Laboratory, Princeton, New Jersey, 08544, USA

Electron dynamics in a beating-electrostatic-wave-powered magnetic null thruster are explored in order to determine a mechanism that allows for electrons to exit the thruster. Electron behavior inside of the region of magnetic field reversal is shown to be different from that of the ions due to the relative difference in characteristic Larmor orbit size compared to the characteristic size of the null. The analysis led to a thruster design criterion stated in terms of the relative sizes of electron and ion Larmor radii. An analytical expression for the fraction of electrons which experience adverse reverse drifting orbits inside the null is derived. Additionally, the expected drift velocity of forward moving electrons within the null region is calculated. A scaling relationship for the rate at which electrons enter the null region through collisional processes is also developed. These three parameters are all shown to be dependent on a single non-dimensional ratio of the electron thermal Larmor radius to the size of the null region.

Nomenclature

\[ \begin{align*}
\omega & \quad \text{Angular frequencies} \\
\rho & \quad \text{Larmor radius} \\
m & \quad \text{Mass} \\
T & \quad \text{Temperature} \\
k & \quad \text{Boltzmann's Constant} \\
B & \quad \text{Magnetic Field} \\
A & \quad \text{Magnetic vector potential} \\
\delta & \quad \text{Magnetic null region size} \\
q & \quad \text{Electric charge} \\
t, \tau & \quad \text{Time and normalized time} \\
P_{X,Y} & \quad \text{Canonical momentum} \\
h, H & \quad \text{Hamiltonian and normalized Hamiltonian} \\
v & \quad \text{Velocity} \\
R & \quad \text{Non-dimensional Ratio} \\
D & \quad \text{Diffusion coefficient} \\
\nu & \quad \text{Collision frequency} \\
n & \quad \text{Density} \\
V & \quad \text{Volume} \\
K & \quad \text{Rate} \\
\_i, e & \quad \text{Ion, electron} \\
c & \quad \text{Cyclotron} \\
g_c & \quad \text{Guiding Center of Cyclotron Motion} \\
th & \quad \text{Thermal} \\
\perp & \quad \text{Perpendicular to } B\text{-field}
\end{align*} \]

*Graduate Student, Mechanical and Aerospace Engineering Department, Research Assistant.
†Chief Scientist, EPPDyL; Professor, Applied Physics Group; AIAA Fellow
I. Introduction

A recent paper by Jorns and Choueiri\textsuperscript{1} proposed a new thruster concept that relies on beating electrostatic waves to energize ions, and a magnetic field reversal to direct these ions linearly out of the thruster. Beating wave ion acceleration is a mechanism for non-resonant acceleration of low energy ions that has its roots in experimental observations in the ionosphere in the 1970s.\textsuperscript{2,3} It is well known that vigorous acceleration of a charged particle can occur when the particle’s velocity is resonant with the nonlinearly broaden phase velocity of a parent wave. However, observations showed low-energy magnetized ions that are non-resonant with the ambient electrostatic waves being accelerated to much higher energies in a region of electrostatic wave turbulence.\textsuperscript{2,3} Benisti proposed the following beating wave mechanism for this non-resonant effect in 1998,\textsuperscript{4,5} wherein two electrostatic waves with frequencies $\omega_1$ and $\omega_2$ interact with a magnetized particle with cyclotron frequency $\omega_c$ such that:

$$\omega_2 - \omega_1 = n\omega_{ic}. \tag{1}$$

Spektor and Choueiri expanded the picture by deriving the necessary and sufficient conditions for single ion acceleration subjected to beating electrostatic waves.\textsuperscript{6} They showed that the initial Hamiltonian of the system, irrespective of the ion’s initial velocity, must be within certain bounds for acceleration to occur. A subset of initially slow moving ions can be coherently accelerated by the beating wave effect until they reach a threshold value. Above this threshold, ions can then experience a stochastic acceleration mechanism, similar to that developed by Karney for single electrostatic waves,\textsuperscript{7} where ions receive periodic “kicks” in energy. Jorns and Choueiri further elucidated the phenomenon\textsuperscript{8,9} and showed that beating waves can be used to target a larger fraction of the ion population for heating than in the single electrostatic wave scenario.

Some work on beating waves has focused on heating an ion ensemble for propulsion purposes.\textsuperscript{3} However, Jorns and Choueiri’s proposed thruster concept uses beating electrostatic waves to target individual ions for acceleration (as opposed to plasma heating). Accelerated ions then interact with a linear magnetic field reversal to direct energized ions out of the thruster.\textsuperscript{1} This thruster relies on the stochastic acceleration mode of the beating wave ion acceleration mechanism to preferentially push ions towards the region of magnetic reversal as the ion is being accelerated. The characteristic ion trajectories about the magnetic null then propagate preferentially forward out of the thruster. Jorns and Choueiri calculated the ion dynamics in such a configuration and showed the potential for directed ion motion. Gardineer et al further showed through numerical simulations that the overall dynamics can generate ion exhaust velocities on the order of 10 km/s for typical beating wave parameters currently used in laboratory experiments.\textsuperscript{10}

The proposed thruster has certain intrinsic advantages that deserve further study. The design is inherently steady-state and electrodeless, which allows the thruster to avoid lifetime limitations of other electric devices. Additionally, because the thrust mechanism occurs across magnetic field lines, the proposed device avoids magnetic detachment issues, which can harm efficiency.

However, while previous work on the beating wave magnetic null thruster has demonstrated the possibility for significant ion acceleration, that work has so far ignored the question of the electron dynamics in such a thruster. Since the inherent thrust mechanism works across the magnetic field, electrons cannot simply be dragged out of the thruster by the accelerated ions. The particle dynamics in the magnetic null topology explored by Jorns and Choueiri\textsuperscript{1} and Gardineer et al\textsuperscript{11} focused on ion ensembles with characteristic Larmor radii larger than the size of the region of magnetic reversal. While the electron equations of motion are inherently the same as ions, because electrons have smaller Larmor radii, in general, it is necessary to understand the small-orbit particle dynamics in the magnetic null topology. This paper explores the small-orbit electron dynamics and derives equations governing particle motion in a magnetic null in the limit where the characteristic Larmor radius is smaller than the region of magnetic field reversal. This single particle analysis is not self-consistent for the whole thruster, as potential dynamics between electrons and ions have been so far ignored. However, this analysis provides a mechanism through which electrons can leave the thruster along with the ions. Additionally, it gives us insight into the difference between small and large Larmor orbits in a magnetic null topology, which can be used to develop design criterion for such a thruster.

The layout for this paper will proceed as follows. In Section II, we review briefly the beating electrostatic wave magnetic null thruster, as well as present the equations of motion governing single particle dynamics in the magnetic null topology. In Section III, we characterize the electron small-orbit dynamics in the same configuration and within the magnetic null region. In Section IV, we discuss a mechanism through which electrons can pass into the null region through collisional effects. Finally, in Section V, we discuss the differences between ion and electron single particle motion within the magnetic null topology.
II. Beating Wave Magnetic Null Configuration and Background

A three-dimensional representation of the Beating Wave Magnetic Null Thruster design can be seen in Figure 1. The thruster consists of two helmholtz-like, current conducting coils with one each above and below the $y = 0$ plane. The upper coil pair generates a uniform magnetic field in the positive-$\hat{z}$ direction. The lower coil has an opposite polarity and generates an equal magnitude magnetic field in the negative-$\hat{z}$ direction. Between the two coils in the region near the $y = 0$ plane, the magnetic field must reverse its sign, and therefore the magnetic field $B_z = 0$ when $y = 0$. The magnetic topology itself does not add energy to the plasma ensemble, but rather, the null region is used to direct ions out of the thruster in the positive-$\hat{x}$ direction. In order to generate significant thrust, beating electrostatic waves are launched in the positive-$\hat{x}$ direction from antennas along the $x = 0$ plane both above and below the null. The launched waves serve to both accelerate and energize individual ions as well as direct them towards the magnetic null region. To understand this process, we will briefly discuss both the magnetic topology and the beating wave effects.

A. Magnetic Topology and Single Particle Dynamics

We consider the thruster in two dimensions inside the page, with the $\hat{z}$-direction pointing out of the page toward the reader. Far above and below the $\hat{x}$-axis, where $y = 0$, the magnetic field is uniform and constant in magnitude. For $y > 0$, the magnetic field comes out of the page; for $y < 0$, the magnetic field is directed into the page as shown in Figure 2. In the immediate region around the magnetic null, an instantaneous field reversal is not possible, and the field is sloped. We refer to this region where the magnetic field is not constant as the null region. Jorns and Choueiri approximated this topology with the following equation:

$$B = B_0 \tanh \frac{3y}{\delta}$$  \hspace{1cm} (2)

which corresponds to an approximately uniform magnetic field when $|y| > \delta$. In the null region, the magnetic field is sloped such that the gradient of the magnetic field is approximately $3B_0/\delta$. (See Figure 2.) The field reversal is designed to exploit ion gyro motion in this region and direct ions along the magnetic null. For a single particle in this field without the effects of beating electrostatic waves, the Hamiltonian governing the
Figure 2. Magnetic Configuration. On the left, we see the Magnetic field has opposite polarity above and below the \( \hat{x} \)-axis. On the right, the Magnetic field, \( B_z \) normalized by \( B_0 \) is plotted against the normalized coordinate \( Y \). Note that above \( Y = 1 \) the magnetic field is roughly uniform. This corresponds to the location where \( y = \delta \).

The equations of motion is given by:

\[
h = \frac{1}{2m} \left( p - qA \right)^2,
\]

where \( p \) is the canonical momentum, \( q \) is the electric charge of the particle, and \( A \) is the magnetic vector potential. Since \( \nabla \times A = B \), we note that \( A = A_x(y) = \frac{q}{2} B_0 \log \cosh \frac{3y}{\delta} \).

We use the same normalization scheme as Jorns and Choueiri\(^1\) except that we normalize our length scale by \( \delta \) instead of the beating wave wavenumber \( k \).

\[
H = \frac{1}{2} \left( \left[ P_X - \ddot{A}_X \right]^2 + P_Y^2 \right),
\]

where

\[
\tau = \omega_c t \quad H = m \delta^2 \omega_c^2 h \quad X = \frac{x}{\delta} \quad Y = \frac{y}{\delta}
\]

\[
\ddot{A}_X = \frac{q}{m \omega_c \delta} A_x(\delta Y) \quad P_X = \dot{X} + \ddot{A}_X \quad P_Y = \dot{Y}
\]

\[
\ddot{\rho} = \frac{\tau}{\omega_c \delta} = \frac{\rho}{\omega_c \delta}
\]

where \( \omega_c = \frac{|qB_0|}{m} \) is the particle cyclotron frequency and \( ' \) represents differentiation with respect to \( \tau \). \( H \) is a constant of motion and is related to the normalized Larmor radius by \( H = \frac{1}{2} \dot{\rho}^2 \).

Ions that do not intercept the null region experience simple Larmor motion. For ions that do intercept the null region, Jorns and Choueiri determined there were four types of ion drift trajectories\(^1\) for particles with characteristic orbits larger than the size of the magnetic null: linear betatron, forward figure-8, reverse figure-8, and grad-\( B \). The linear betatron and forward figure-8 orbits drift in the positive-\( \hat{x} \) direction, while the reverse figure-8 and grad-\( B \) trajectories drift backwards. These trajectories are dependent on magnitude of \( \ddot{\rho} \) and the guiding center location, \( Y_{gc} \) and are shown in Figure 3. The Beating Electrostatic Wave Magnetic Null configuration attempts to exploit the positive drifting orbits to generate axial ion motion in the positive-\( \hat{x} \) direction. Therefore, the backward drifting ion orbits are harmful for thrust generation. These harmful orbits are a result of the non-ideal field reversal. For example, an instantaneous step function field reversal would yield no reverse drifting ions. Instead only the forward drifting linear betatron and figure-8 orbits would exist.\(^1\) Therefore, we can see that one important criterion for a magnetic null configured thruster is that ion Larmor orbits be larger than the characteristic length scale of the field reversal.

\[
\ddot{\rho}_i > 1
\]

We will see later that imposing the opposite criterion for electrons, where there gyro radius is much smaller than the null region, will help electrons escape forward in the positive-\( \hat{x} \) direction as well.
B. Beating Electrostatic Waves

The addition of beating electrostatic waves serves two important functions in this thruster configuration. The most obvious purpose is to direct energy into individual ions, a process characterized for individual ion motion by Spektor and Choueiri. As energy is transferred into individual ions, the basic acceleration effect occurs in two distinct mechanisms. The first is a non-resonant process which can steadily and coherently increase the ion velocity until it reaches a threshold value. At this point, an ion has sufficient velocity that it can be subjected to vigorous acceleration by a resonant process where ions stochastically receive periodic “kicks” in energy.

The second function is to direct ions from the large regions of uniform magnetic fields towards the magnetic null region. This process occurs due to the stochastic acceleration mechanism. While in the stochastic regime, as an ion’s energy increases, it is pushed towards the $y = 0$ magnetic null. Mathematically speaking, Jorns and Choueiri derived the following relationship between the ion guiding center of motion and the energy increase:

$$\langle |y_{gc}| \rangle = |y_{gca}| - \frac{1}{2v_{ph}\omega_{ic}} (v_i^2 - v_{i0}^2),$$

(6)

where $y_{gca}$ is the initial guiding center location, $v_{ph}$ is the Beating Wave phase velocity, $\omega_{ic}$ is the ion cyclotron frequency, and $v_{i0}$ is the ion threshold velocity, which is the velocity at which an ion begins to experience the stochastic mode. Clearly, as the ion velocity increases, its guiding center moves towards $y = 0$.

While Jorns and Choueiri characterized which ions can be targeted for direct acceleration via beating electrostatic waves, further simulations by Gardineer, et al have demonstrated that beating electrostatic waves can be used to generate significant directed ion acceleration with exit velocities of at least 10 km/s.

C. The Electron Question

The ion motion previously explored and briefly described above does indeed result in a net axial translation of ions which can produce thrust. However, to this point, all analysis has focused solely on ion
motion, which is assumed to have characteristic Larmor orbits larger than the null region. Since electrons are highly magnetized, there is no immediately obvious way in which an electric field would pull electrons out of the thruster along with the ion motion. In order to demonstrate the feasibility of this design, electron dynamics must be understood, and a mechanism for electron motion out of the thruster must be identified.

Electrons experience the same equations of motion as ions, but since electrons have an opposite charge compared to ions, each of their orbits occurs in the reverse direction from the corresponding ion orbit. Thus, the “linear betatron” and “forward figure-8” orbits drift in the negative-\( \hat{x} \) direction, and the “grad-B” and “reverse figure-8” orbits drift in the positive \( \hat{x} \) direction. One might expect that electrons primarily drift towards the beating wave antenna, instead of out of the thruster with the ions. However, electrons typically have gyro orbits much smaller than ions. Therefore, it is important to understand the dynamics of these particles small orbits that are entirely within the null region. In the next Section, we will show that the dominant effect for electrons with characteristically small gyro radii is to drift in the positive-\( \hat{x} \) direction with the ion motion.

**III. Electron Dynamics within the Null Region**

To address the electron dynamics in the magnetic null thruster, we first explore those dynamics solely within the magnetic null region. Typical electron orbits within the null region are shown in Figure 4. These orbits were calculated with the same normalized equations of motion as the above ion orbits in Figure 3, but for particles with characteristically small gyro radii. That is,

\[
\bar{\rho}_e << 1.
\]

The four orbits in Figure 4 correspond to the four types of ion orbits, except that the oppositely charged electrons experience reversed drifts compared to ion motion. The rest of this Section demonstrates that for an electron thermal distribution such that

\[
\bar{\rho}_e = \frac{m_e v_{e,th}}{qB_0} << 1,
\]

electrons will primarily drift in forward in the positive-\( \hat{x} \) direction.

Unlike ions, in the small orbit limit, most electrons spend all of their time either completely within the null region or completely outside of the null region. Moreover, only those electrons with orbits which remain close to the null plane at \( y = 0 \) are able to physically cross the magnetic null. These orbits no longer have well-defined guiding centers of motion and correspond to three orbits previously shown for ions: linear betatron, forward figure-8, and reverse figure-8. However, we note that the negatively charged electrons drift in the opposite direction of the ions for each of these orbit types. On the other hand, electrons that remain far enough from the null plane but still within the null region maintain their gyro motion and experience grad-B drifts forward in the positive-\( \hat{x} \) direction.

In the case of ions, having large gyro radii limited the effect of the magnetic gradient in the null region, resulting in less backward-drifting orbits. For electrons, having a smaller gyro radius amplifies the effect of the magnetic gradient in the null region. In fact, most electrons in the null region will experience simple grad-B drifts. In order to see this effect, we calculate the \( y \)-coordinate where the electrons stop experiencing grad-B drifts. This occurs when the \( y \)-coordinate of the guiding center of motion is equal to the gyro radius of a given electron, thus allowing for the electron to cross the null plane:

\[
y_{gc} = \frac{m_e v_e}{qB(y_{gc})} = \frac{m_e v_e}{qB_0 \tanh 3y_{gc}/\delta}.
\]

Near \( y = 0 \), hyperbolic tangent can be approximated by \( \tanh x = x \). Solving through Eq 9, we determine see that

\[
y_b = \sqrt{\frac{m_e v_e \delta}{3qB_0}}.
\]

We use \( y_b \) to define this critical location, which we see is proportional to the geometric mean of the null region size, \( \delta \), and the electron gyro radius.

With this information, we can now determine what fraction of electrons in the null region cross the null plane. To do this, we integrate over the phase space coordinates \( y_{gc} \) and \( v_\perp \) with a Maxwellian electron
temperature distribution and electrons evenly distributed in guiding center location. The fraction of electrons which cross the null plane is given by:

$$\int_0^\infty \int_{y_b}^0 \frac{m_e v}{\delta kT_e} e^{-\frac{m_e v^2}{2\delta kT_e}} dydv,$$

where \( y_b = \sqrt{\frac{m_e v^3}{3qB_0}} \). This corresponds to an electron fraction crossing the null of:

$$f_{e,\text{null}} = \frac{1}{\sqrt{3}} \left[ \frac{5}{4} \right] \left( \frac{2kT_e m_e}{qB_0 \delta} \right)^{\frac{3}{2}} \approx 0.523 \sqrt{\frac{v_{e,th}}{\omega_{ec}\delta}}.$$

For typical beating wave plasma values of \( T_e = 3 \text{ eV}, B_0 = 500 \text{ Gauss} \) and assuming a null region size of \( \delta = 1 \text{ cm} \), this corresponds to 5% of the electrons within the null region that will cross the null plane. The rest will be in grad-B orbits, which are always forward drifting.

It is also important to know how fast these electrons drift while inside the null region. Grad-B drift velocities for electrons are easily calculated from the well-known formula:

$$v_{\nabla B} = \frac{mv^2}{2qB} \frac{\nabla B \times B}{B^2}.$$

We can calculate the expected value of the drift velocities of these grad-B drifting electrons by again integrating over phase space for all electrons with guiding centers above the critical value \( y_b \),

$$\langle v_d \rangle = \int_0^\infty \int_{y_b}^\delta v_{\nabla B} \frac{m_e v}{\delta kT_e} e^{-\frac{m_e v^2}{2\delta kT_e}} dydv.$$

We introduce a new constant, \( R \),

$$R = \bar{v}_{e,th} = \frac{\rho_{e,th}}{\delta} = \frac{v_{e,th}}{\omega_{ec}\delta}.$$

Figure 4. Plots of electron trajectories within the null region with \( H = 5 \times 10^{-5} \) which corresponds to \( \bar{v} = 0.01 \). \( X \) and \( Y \) are normalized by the size of the null region, \( \delta \). The top two trajectories in red correspond to the ion Grad-B drifts and Reverse Figure-8 drifts, but for electrons, they are forward drifting orbits. The bottom two trajectories in blue correspond to the ion Forward Figure-8 and Linear Betatron drifts, but for electrons, they are reverse drifting orbits.
which corresponds physically to the normalized thermal Larmor radius, and leave the full algebraic derivation in the Appendix, which gives us an expected value for the drift velocity that is a function of the electron thermal velocity, \( v_{th} \) and our new constant, \( R \).

\[
\langle v_d \rangle = v_{th} R g(R) = v_{th} R \left[ \int_0^\infty s^3 \coth(\sqrt{3}Rs) e^{-s^2} ds - \frac{1}{2} \right]
\]  

(16)

A plot of \( R g(R) \) is shown in Figure 5. We now have an expression for the expected electron velocity flow along the null region in terms of the non-dimensional \( R \). Since \( R \) is a ratio between the average electron gyro radius and the size of the null region, we expect it to be much less than unity. While increasing \( R \) does result in faster \( \nabla B \) drift motion, it also results in more electrons crossing the null and experiencing reverse drifts. Even for larger values of \( R \), however, the maximum value of the function \( R g(R) \) is less than .05.

For a typical plasma currently generate in beating wave experiment, \( T_e = 3 \text{ eV} \) and \( B_0 = 500 \text{ Gauss} \). Assuming a null region size of \( s = 1 \text{ cm} \), we calculate an expected drift velocity of \( \sim 20 \text{ km/s} \). Interestingly, recalling that Gardineer et al simulated expected ion exit velocities of \( \sim 10 \text{ km/s} \), it possible that an efficient beating electrostatic wave magnetic null thruster could be tuned such that ions and electron drift forward with the same characteristic velocities. However, it would still be necessary to ensure that flux of electrons and ions to the null is equal.

**IV. Electron Motion Into the Null Region**

So far, we have explored the electron motion inside of the null without concerning ourselves with electron motion outside of the null. However, unlike ion motion into the null region, which is controlled by beating wave effects, electrons are not preferentially pushed towards the null or into the null region. Without such a mechanism, electrons will remain stuck in Larmor orbits in the region of uniform magnetic field. One mechanism through which electrons can enter the null is diffusion through collisional effects. The magnetization of electrons will impede this diffusion process, and we calculate the rate at which electrons enter the null region using the cross-field diffusion coefficient is given by:

\[
D_\perp = D_e \frac{1}{1 + (\omega_{ce}/\nu_e)^2},
\]  

(17)

where \( D_e \) is the field-free diffusion coefficient, \( \omega_{ce} \) is the electron cyclotron frequency, and \( \nu_e \) is the electron collision frequency. The field-free coefficient, \( D_e \), scales with \( kT_e/(m_e\nu_e) \). So we can write:

\[
D_\perp = \frac{kT_e}{m_e} \frac{\nu_e}{\nu_e^2 + \omega_{ce}^2}.
\]  

(18)
However, since electrons will be highly magnetized in our configuration, we know that $\omega_{ce} \gg \nu_e$. Therefore, we can simplify the scaling relationship to

$$D_\perp \approx \frac{kT_e \nu_e}{m_e \omega_{ce}^2}. \quad (19)$$

The flux of electrons into the null region is given by:

$$\Gamma_e \approx D_\perp \nabla n_e = \frac{kT_e \nu_e}{m_e \omega_{ce}^2} \nabla n_e. \quad (20)$$

In the case where the null region is depleted compared to the region of uniform magnetic field, the density gradient will scale as $n_{eu}/\delta$, where $n_{eu}$ is the electron density in the region of uniform magnetic field, which gives us a scaling relation for the electron flux into the null as:

$$\Gamma_{e,in} \approx \frac{kT_e \nu_e}{m_e \omega_{ce}^2} \frac{n_{eu}}{\delta}. \quad (21)$$

In steady state, we know that $\Gamma_{e,in} = \Gamma_{e,out} = n_{en}(v_d)$, where $n_{en}$ is the electron density in the null region. If the null is depleted compared to the region of uniform field, then $n_{en} << n_{eu}$, which means that our assumption holds when $(v_d) >> \delta R^2 \nu_e$.

The rate at which electrons enter the null region is equal to the flux, $\Gamma_e$ times the area of the thruster in the $\hat{x} - \hat{z}$-plane. After some algebra, we can then write the scaling for the rate at which electrons flow into the null:

$$K_e \approx n_e V_{null} R^2 \nu_e. \quad (22)$$

where $V_{null}$ is the total volume of the null region, $n_e$ is the density outside the null, and $R = v_{th}/(\omega_e \delta)$ is our same non-dimensional coefficient from Sec III. Since the electron flow into and out of the null region will be equal in steady state, this represents the characteristic rate at which electrons can be ejected from the thruster, most of which will be in the positive-$\hat{x}$ direction.

V. Summary and Concluding Remarks

In this paper, we discuss the relationship between a particle’s characteristic gyro radius and the size of the magnetic null region. In particular, we see that particles with Larmor orbits much smaller than the null region will experience net drift motion of the opposite type to those with Larmor orbits larger than the null region. For smaller orbits, grad-$B$ drifts dominate, while linear betatron and forward figure-8 orbits dominate in the large orbit regime. These different drifts have opposite directionality.

Additionally, electrons and ions have opposite charge, which also reverses the directionality of each type of drift. As a result, by allowing electrons to have characteristically small gyro orbits and keeping ion orbits larger than the null region, we can ensure that both types of particles experience predominantly forward drifting orbits. Mathematically speaking:

$$\bar{\rho}_{e,i} \ll 1 < \bar{\rho}_i \quad (23)$$

When this criterion is satisfied, electrons primarily drift forward while in the null region through grad-$B$ drifts and ions primarily drift forward as the intersect the null region in linear betatron and forward figure-8 drifts. This type of characteristic inequality is similar to a Hall thruster, which usually obeys $\Omega_e \gg 1 > \Omega_i$, where $\Omega_{e,i}$ is the Hall parameter. This type of inequality criterion implies that the ions and electrons should experience the null region in a fundamentally different way in a magnetic null topology that attempts to propel both particles forward in the positive-$\hat{x}$ direction. Additionally, it serves as a design criterion for the size of the null region, $\delta$.

Ion motion subjected to beating electrostatic waves has previously been characterized from the single particle perspective with typical orbits larger than the size of the null region. However, the small orbit effects that the electron ensemble experiences had so far been ignored. We have derived three scaling relationships governing the independent motion of electrons in a magnetic null configured beating electrostatic wave thruster: An upper bound on the fraction of electrons in the null region which experience reverse drifting orbits,

$$f_{rev} \approx .523 \sqrt{\frac{\Omega_{e,th}}{\omega_e \delta}} = .523 \sqrt{\tilde{R}}, \quad (24)$$
the average speed at which electrons drift forward while within the null region,

\[ \langle v_d \rangle = v_{th} \ast R \left[ \int_0^\infty s^3 \coth(\sqrt{3R}s)e^{-s^2} ds - \frac{1}{2} \right], \]  

(25)

and a scaling for the rate and which electrons enter the null region through collisional effects,

\[ K_e \sim n_e V_{null} R^2 v_e. \]  

(26)

Interestingly, each effect is dependent on a non-dimensional parameter, \( R \), which is equal to the ratio of the electron thermal Larmor radius \( \rho_{th} = v_{th}/\omega_e \) to the null region size, \( \delta \). This makes intuitive sense because the electron dynamics differ from the ion dynamics due to the relative sizes of their Larmor orbits compared to the null region. One assumption used to calculate these scaling relationships is that the electron gyro motion is much smaller than the size of the null region. Therefore, these equations only hold for values of \( R \) less than unity.

As \( R \) is increased, the fraction of electrons that cross the magnetic null increases. This corresponds directly to an increase in the number of reverse drifting electrons. Increasing \( R \) initially leads to an increase in the average forward drift velocity of the electrons within the null. While there is a upper bound of no more than 5% of the electron thermal velocity, since the electron thermal velocity is large, this can correspond to drift velocities of up to 50 km/s. Increasing \( R \) also results in improved diffusion of electrons into the null region. This also makes sense, as electrons with larger Larmor orbits are effectively less magnetized with respect to the size of the thruster. Overall, while increasing \( R \) can lead to an enhancement of electron motion into the null, as well as an enhancement in electron forward drift speed, it also has the deleterious effect of increasing the electron fraction in the null that experiences backward drifting orbits in the negative-\( \hat{x} \) direction. As a result, there are fundamental limitations on the rate at which electrons can leave the thruster in the positive-\( \hat{x} \) direction through the collisional process.

Finally, we note the mechanism through which small-orbit electrons leave the thruster is fundamentally different than the heating wave mechanism used to direct ions forward out of the thruster. We have already seen that the electron drift velocity can possibly be matched to previously simulated\(^{10} \) ion exhaust velocities. The non-dimensional constant, \( R \), can serve as a tunable parameter to help ensure that electron motion out of the thruster matches the ion motion both in terms of the characteristic electron drift velocity and the rate at which electrons leave the thruster.

**Appendix**

This appendix shows the full derivation of Eq 16. Starting with Eq 14, we have

\[ \langle v_d \rangle = \int_0^\infty \int_{y_b}^\delta v_{\nabla B} \ast \frac{1}{\delta} \frac{m_e v^3}{kT_e} e^{-\frac{m_e v^2}{2kT_e}} dy dv. \]  

(27)

Substituting Eq 2 into Eq 13 we see that:

\[ v_{\nabla B} = \frac{3}{2} \frac{m_e v^2}{qB_0} \csch^2(3y/\delta), \]  

(28)

which gives us:

\[ \langle v_d \rangle = \int_0^\infty \int_{y_b}^\delta \frac{3}{2} \frac{m_e v^3}{qB_0} \delta^2 kT_e \csch^2(3y/\delta) e^{-\frac{m_e v^2}{2kT_e}} dy dv. \]  

(29)

Preforming the substitution, \( y^* = \frac{y}{\delta} \) and \( s = \frac{v}{v_{th}} = v \sqrt{\frac{m_e}{2kT_e}} \), Eq 29 becomes:

\[ \langle v_d \rangle = \int_0^\infty \int_{y_b^*}^1 \frac{3}{2} \frac{v_{th}^2}{\omega_e \delta} s^3 \csch^2(3y^*) e^{-s^2} dy^* ds. \]  

(30)

Note that \( y_b^* = \frac{y_b}{\delta} = \sqrt{\frac{v}{3\omega_e \delta}} = \sqrt{\frac{v_{th}}{3\omega_e \delta}} s \) and recall that \( R = \frac{\nu_h}{\omega_e \delta} \). Eq 30 becomes:

\[ \langle v_d \rangle = v_{th} \ast R \int_0^\infty \int_{y_b^*}^1 \frac{3}{4} \frac{v_{th}^2}{\omega_e \delta} s^3 \csch^2(3y^*) e^{-s^2} dy^* ds. \]  

(31)
Simple integration then yields Eq 16:

$$\langle v_d \rangle = v_{th} \ast R \left[ \int_0^\infty s^3 \coth(\sqrt{3Rs}) e^{-s^2} \, ds - \frac{1}{2} \right]$$

(32)

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References


