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Plasma detachment and momentum transfer in magnetic nozzles

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The nature of momentum transfer and the resulting thrust generation in magnetic nozzles is investigated. First, it is shown analytically using a Green's function formulation that thrust transmission results from the interaction of the magnetic field induced by the currents in the plasma with the current in the applied field coil and is equal and opposite to the integral of the volumetric and surface Lorentz force densities due to the applied magnetic field acting on the plasma. Second, using a two-fluid plasma model, it shown that, contrary to previous belief [Ahedo and Merino, Phys. of Plas., 18(5) 2011], positive thrust production can occur for a detachment mechanism that induces paramagnetic plasma currents, as long as a criterion, which ensures the dominance of the force density due to the diamagnetic current at the plasma-vacuum boundary (which contributes to thrust) over that due to paramagnetic current, which results from the inertial detachment process (and which diminishes thrust), is satisfied. The model also shows that the thrust efficiency suffers with increasing magnetic field divergence and plasma magnetization, which enhance the relative contribution of the paramagnetic current; and that inertial detachment occurs when a hybrid particle of mass $m_H = (m_e M_i)^{1/2}$ becomes demagnetized.

I. Introduction

Simply stated, magnetic nozzles convert the thermal energy of a plasma into directed kinetic energy. This conversion is achieved by means of a strong convergent-divergent magnetic field contoured similarly to the solid walls of a conventional de Laval nozzle.[1] The applied field, typically formed by passing a large current through an electromagnetic coil, confines the plasma and acts as an effective "magnetic wall" through which the thermal plasma expands. The feasibility of using plasma flow along magnetic fields to produce thrust has been questioned, however, due to the tendency of the highly conductive plasma to remain tied to necessarily closed magnetic field lines. Efficient detachment of the plasma from the magnetic nozzle thus becomes paramount for the potential application of such concepts to space propulsion.[2–9]

Several physical mechanisms have been proposed to explain plasma detachment from magnetic nozzles. Cohen was the first to propose that the plasma detachment problem may be avoided if the electron temperature within the expanding plume decreased at a rate that would allow significant three-body recombination of the ions into neutral particles, thus producing an unmagnetized plume.[4] Moses, on the other hand, suggested that detachment could occur via resistive diffusion of the plasma across the applied magnetic field lines.[2] Many plasmas of interest for propulsion applications are of sufficiently low density and high electron temperature to neglect collisional effects within the expanding plasma. In this limit, Hooper demonstrated that detachment is theoretically possible for the plasma to separate from the magnetic field due to its own inertia.[3] Finally, Arefiev and Breizman claim that detachment will occur as induced azimuthal plasma currents drag the magnetic field along with the flow.[5]

The generation of useful thrust requires the transmission of momentum from the accelerated propellant to the structure of the magnetic nozzle. Thrust transmission in a conventional rocket is achieved through pressure forces acting on the solid surfaces of the propellant injector, combustor, and nozzle (Figure 1(a)). A magnetic rocket, on the other hand, is not guaranteed to have any solid surfaces in contact with the

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propellant. Thus, the following question is raised: *How is momentum transmitted from the expanding plasma to the structure of a magnetic nozzle?*

The issue of thrust generation within the context of plasma detachment from magnetic nozzles was recently investigated by Ahedo and Merino.[9] They use a two-dimensional model to characterize the expansion of a cold ion, hot electron plasma through a diverging magnetic field.[10] Using a standard control volume analysis, they show that the change in momentum within the flow is equal to the sum of the volumetric and surface Lorentz forces acting within the plasma and at the plasma-vacuum interface, respectively. From this result, they conclude that momentum transfer to the magnetic nozzle must be achieved through the mutual interaction between the currents within the nozzle coil and the induced plasma currents (Figure 1(b)). Specifically, azimuthal plasma currents that develop in the direction opposing the nozzle current repel the nozzle and produce positive thrust gain in the divergent section of the nozzle. Azimuthal currents in this direction decrease the magnetic field, and therefore are referred to as diamagnetic. Alternatively, plasma currents flowing in the same direction as the nozzle current are paramagnetic and attract the nozzle coil, thus negating thrust.

The plasma-nozzle interaction described above leads Ahedo and Moreno to claim that positive thrust gain requires diamagnetic azimuthal plasma currents. Furthermore, he shows that the processes describing resistive detachment, inertial detachment, and magnetic detachment all infer the presence of paramagnetic currents. However, the results that he presents are not fully consistent as they do not include the influence of collisionality, cross-field electron transport, or induced magnetic fields. As a result, the relation between plasma detachment, paramagnetic currents, and thrust transmission is still unclear.

The question then remains: *how do currents induced through plasma detachment influence momentum transfer in magnetic nozzles?* The issue of momentum transfer and its relation to *inertial* plasma detachment will be our central focus in this paper. First, we provide a rigorous proof of the claim that the momentum transfer for magnetic nozzles is due to the repulsion of the applied field coil of the nozzle from currents induced in the plasma. With this result, we propose a requirement less restrictive than that of Ahedo and Merino on the nature of the currents induced by plasma detachment. We then use a previously developed two-fluid model[7] for the expansion of a magnetic nozzle plasma to prove that thrust gain may still occur for plasma detachment mechanisms that generate paramagnetic currents. Furthermore, we find that the drag attributed to these paramagnetic currents scales with the divergence of the exhaust plume. Finally, we analyze the magnetization of the plasma at the point of separation from the magnetic field to provide some insight into the fundamental physics of inertial detachment.

II. Thrust Transmission

The main goal of this section is to prove the claim that momentum transfer in magnetic nozzles manifests from the interaction between currents within the nozzle coil and induced plasma currents. Taking the reference frame of the nozzle coil and considering the force transmitted to the coil by volumetric and surface currents within the plasma downstream the nozzle throat, we ultimately arrive at the same thrust expression that Ahedo and Merino derived using a control volume analysis. We then return to the control volume derivation to expand upon their result to include the influence of induced magnetic fields.

We begin with the assumption that the applied magnetic field is generated by a single current loop of infinitesimal cross-section with strength, I. For this analysis, we ignore all other magnetic coils within the source region ($z < 0$ in Figure 1(b)). The force experienced by our current loop, or "nozzle coil," is given by the integral of the Lorentz force acting over its circumference

$$\mathbf{F}^c = \oint_{coil} \mathbf{I} \times \mathbf{B}^{(i)} dl, \tag{1}$$

where $\mathbf{B}^{(i)}$ is the induced magnetic field vector due to all other currents within our domain. Clearly, $\mathbf{I} = I\hat{\mathbf{e}}_\theta$, where $\hat{\mathbf{e}}_\theta$ is the unit vector in the azimuthal direction. Therefore, the goal becomes to express the induced magnetic field vector in terms of the surface and volumetric plasma currents.

It has been shown that the axisymmetric nature of a magnetic nozzle inhibits the formation of a significant azimuthal magnetic field component[3] and allows us to ignore any contributions due to currents in the radial and axial direction. This property permits the description of the induced magnetic field in terms of a scalar flux function, ψ,

$$\mathbf{B}^{(i)} = \frac{1}{r}\left(\hat{\mathbf{e}}_\theta \times \nabla\psi\right). \tag{2}$$

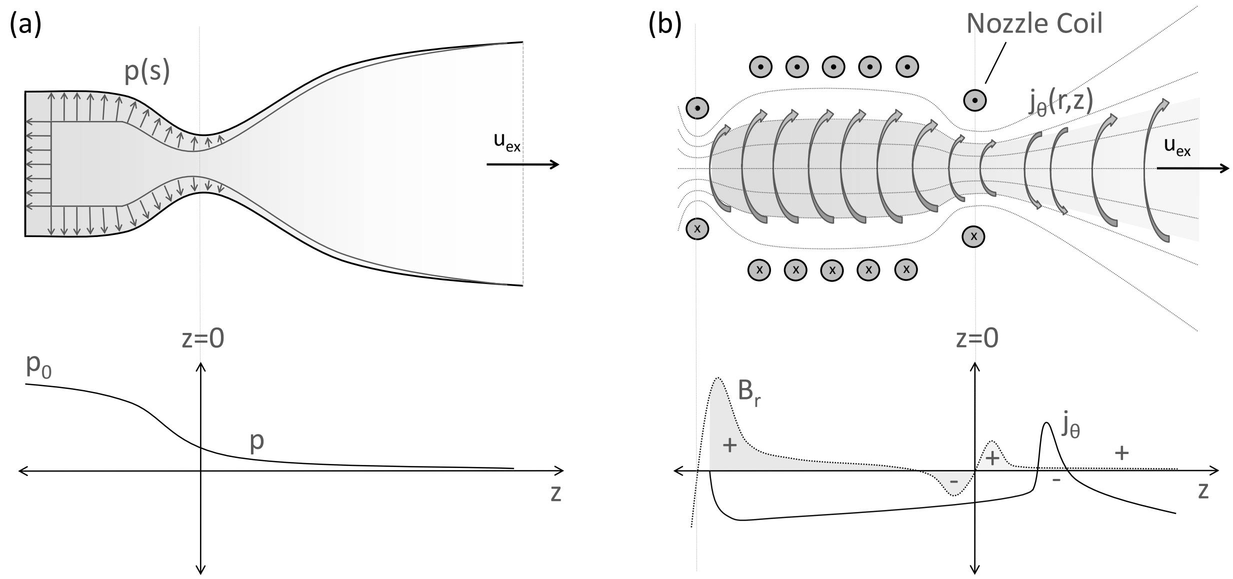

Figure 1. Momentum transmission illustrated by: (a) The pressure distribution, $p(s)$, on the interior surfaces of a conventional rocket. (b) The interaction between the applied magnetic field topology and azimuthal current density, $j_\theta(r,z)$, within a magnetic nozzle. Negative (diamagnetic) currents in the divergent region aid thrust (+) while positive (paramagnetic) currents inhibit thrust (-).

Substitution of Eq.(2) into Ampere's law for a steady-state flow yields the following relationship between ψ and the induced plasma currents, j_θ:

$$L\psi = \frac{1}{r}\nabla^2\psi - \frac{1}{r^2}\frac{\partial\psi}{\partial r} = \mu_0 j_\theta. \tag{3}$$

Here, we have defined the operator, L, to correspond to the left-hand side of Ampere's law.

It is possible to define a Green's function, G, for operator L, such that $LG(\mathbf{r},\mathbf{r'}) = \delta(\mathbf{r}-\mathbf{r'})$, where δ is the Dirac delta function. The Green's function takes the form[11]

$$G(\mathbf{r},\mathbf{r'}) = \frac{1}{2\pi}\frac{\sqrt{rr'}}{k}\left[\left(2-k^2\right)K\left(k^2\right) - 2E\left(k^2\right)\right], \tag{4}$$

with

$$k^2 = \frac{4rr'}{\left(r+r'\right)^2 + \left(z-z'\right)^2}. \tag{5}$$

Furthermore, we define the inner product between two scalar functions, f and g, as $\langle f,g\rangle = \int fg dA$. Here, dA represents the differential area within the $r-z$ plane.

Using Eqs. (3) and (4) in conjunction with the inner product yields,

$$\langle L\psi, G\rangle - \langle \psi, LG\rangle = \int \nabla\cdot\left(\frac{G}{r}\nabla\psi - \frac{\psi}{r}\nabla G\right)dA. \tag{6}$$

where the term on the right-hand side tends to zero for the domain $r\in[0,\infty)$ and $z\in(-\infty,\infty)$. We thus arrive at an expression for the induced flux function in terms of currents developed within the plasma:

$$\psi = \langle L\psi, G\rangle = \mu_0\langle j_\theta, G\rangle. \tag{7}$$

Physically, Eq. (7) represents the sum of the differential flux contributions at location $\mathbf{r}$ due to an infinitesimal current loop of strength $j_\theta dr'dz'$ located at $\mathbf{r'}$.

The thrust generated by the expanding plasma is equivalent to the axial component of Eq. (1). Denoting $\mathbf{r}_c$ as the radius of the coil and $r_c = \mathbf{r}_c\cdot\hat{\mathbf{e}}_r$, we may express the force on the coil in the axial direction as

$$F_z^c = -2\pi r_c I B_r^{(i)}(\mathbf{r}_c), \tag{8}$$

where

$$B_r^{(i)}(\mathbf{r}) = -\frac{\mu_0}{r}\int j_\theta(\mathbf{r'})\, G_z(\mathbf{r},\mathbf{r'})dA' - \frac{\mu_0}{r}\int J_\theta(\mathbf{r'})\, G_z(\mathbf{r},\mathbf{r'})ds', \tag{9}$$

is the radial component of the induced magnetic field as obtained from Eqs.(2) and (7). j_θ and J_θ are the volumetric and surface current densities within the plasma, respectively. G_z represents the partial derivative of the Green's function with respect to z, and ds' is a differential length element along the plasma-vacuum edge.

We now use Eq.(9) and the symmetry of the Green's function derivative, $G_z(\mathbf{r},\mathbf{r'}) = -G_z(\mathbf{r'},\mathbf{r})$, to rewrite Eq.(8) as

$$F_z^c = \int j_\theta(\mathbf{r'})\left[-\frac{\mu_0 I}{r'}G_z(\mathbf{r'},\mathbf{r}_c)\right]2\pi r' dA' + \int J_\theta(\mathbf{r'})\left[-\frac{\mu_0 I}{r'}G_z(\mathbf{r'},\mathbf{r}_c)\right]2\pi r' ds'. \tag{10}$$

Furthermore, we recognize the term in the square brackets as the applied magnetic field,

$$B_r^{(a)}(\mathbf{r}) = -\frac{\mu_0}{r} I G_z(\mathbf{r},\mathbf{r}_c). \tag{11}$$

Due to axisymmetry, we transform the surface and line integrals into volume ($dV' = 2\pi r' dA'$) and surface ($dS' = 2\pi r' ds'$) integrals, respectively. The resulting axial force on the applied field coil is thus,

$$F_z^c = \int j_\theta(\mathbf{r'})\, B_r^{(a)}(\mathbf{r'})dV' + \int J_\theta(\mathbf{r'})\, B_r^{(a)}(\mathbf{r'})dS' = -F_{L,V}^{(a)} - F_{L,S}^{(a)}, \tag{12}$$

which is equal and opposite to the integral over all space of the volumetric, $F_{L,V}^{(a)}$, and surface, $F_{L,S}^{(a)}$, Lorentz forces acting on the plasma due to the *applied (a)* magnetic field. Thus, we recover the result obtained by Ahedo and Moreno with the exception that a distinction must be made between the applied magnetic field and the total magnetic field.

With this understanding of momentum transmission for magnetic nozzles, we may now proceed with the classic control volume derivation for thrust. Consider the control volume $CV_1 : r \in [0,\infty);\ z \in [0, z^*)$, with $z = 0$ denoting the location of the nozzle throat and z^* any location far enough into the plume that the integrals asymptote. Eq.(12) allows us to express the thrust as

$$F = F_0 + F_{L,V}^{(a)} + F_{L,S}^{(a)}. \tag{13}$$

Here, F_0 is the momentum flux at the nozzle throat, and $F_{L,V}^{(a)}$ and $F_{L,S}^{(a)}$ are integrals of the *applied* volumetric and surface Lorentz forces downstream the throat, respectively.

We now take the sum of the ion and electron momentum equations and integrate over the control volume, $CV_2 : r \in [0,\infty);\ z \in (-\infty, z^*)$. We neglect the electron mass compared to the ion mass ($m_e << M_i$), set the plasma density equal to the ion density ($\rho = \rho_i$), and assume the plasma pressure is isotropic and is the sum of the ion and electron pressures ($p = p_e + p_i$). Eq.(13) then allows us to express the thrust as

$$F = F_m + F_p - F_{L,V}^{(i)} - F_{L,S}^{(i)}, \tag{14}$$

where

$$F_m = 2\pi\int \rho u_z^2 r dr, \quad F_p = 2\pi\int p r dr, \tag{15}$$

$$F_{L,V}^{(i)} = -2\pi\int j_\theta B_r^{(i)} r dA, \quad F_{L,S}^{(i)} = -2\pi\int J_\theta B_r^{(i)} r ds, \tag{16}$$

are the momentum, pressure, and *induced (i)* volumetric and surface Lorentz force contributions to the thrust, respectively. The integrals in Eq.(15) are evaluate at $z = z^*$, while the integrals in Eq.(16) are evaluated throughout the plasma upstream from the point $z = z^*$.

We recognize Eq.(14) as the thrust equation for a conventional nozzle with two additional terms related to the interaction between the plasma currents and the magnetic fields that they induce. Using Ampere's law to express the current densities in terms of the induced magnetic field, Eq.(16) gives rise to the thrust contribution of what is typically regarded as the magnetic pressure. The individual contributions of $F_{L,V}^{(i)}$ and $F_{L,S}^{(i)}$ relative to either F_m or F_p scale linearly with β, where β is defined as the ratio of the plasma energy to the magnetic field energy within the nozzle source. The thrust equation thus becomes equivalent to that of a conventional nozzle in the low-β limit where the momentum carried by induced magnetic fields is negligible compared to the momentum in the flowing plasma.

Finally, physical insight may be gained by considering a third control volume defined as the shell of infinitesimal thickness containing the plasma-vacuum boundary. Integrating the axial component of the momentum equation over this control volume yields

$$\int p\hat{\mathbf{e}}_z \cdot \mathbf{dS} = F_{sp} = F_{L,S}^{(a)} + F_{L,S}^{(i)}. \tag{17}$$

Eq.(17) describes the confinement of the expanding plasma in the axial direction, which is a balance between the plasma pressure and the Lorentz forces within the induced diamagnetic current layer developed at the plasma edge.[10]

A parallel may be drawn between momentum transmission in conventional and magnetic nozzles. In a conventional nozzle, thrust is the result of the pressure distribution along the inner surfaces of the nozzle. A magnetic nozzle, on the other hand, may not have any solid surfaces to balance the pressure of the expanding plasma. Rather, surface currents are induced that effectively act as a "magnetic wall" that both confines the expanding plasma, Eq.(17), and transmits the momentum from the plasma to the applied field coil through their mutual interaction, Eq.(12).

III. Plasma Detachment Model

We proved in the previous section that the mutual interaction between the nozzle coil current and induced plasma currents transfers momentum from the expanding plasma to the thruster. Given that the radial component of the applied magnetic field in the divergent section of the nozzle is always positive, Eq. (12) implies that diamagnetic currents($j_\theta < 0$, $J_\theta < 0$) enhance thrust while paramagnetic currents ($j_\theta > 0$, $J_\theta > 0$) degrade thrust. We then must ask: *how do currents induced through plasma detachment influence momentum transfer in magnetic nozzles?* To answer this question, we consider the results from a computer model of plasma expansion and detachment from a magnetic nozzle. This model is described here.

In a previous study[7] on the relationship between inertial plasma detachment and detachment via induced magnetic fields, we introduced a two-fluid model for the expansion of a cold-ion, hot-electron plasma through an applied dipole magnetic field. The plasma is considered uniform at the throat and collisionless throughout the entire plume. Local ambipolarity is assumed to hold everywhere whereby the electrons and ions follow the same trajectory in the meridional (r-z) plane. More on this assumption will be included later. Finally, acceleration is assumed to be quasi-one-dimensional and radial thermal expansion is ignored compared to expansion due to the magnetic field. This last assumption is valid so long as, within the acceleration region, the thermal energy in the plasma is much less than the magnetic energy stored in the fields.

These assumptions allow us to integrate the ion momentum equation to solve for the trajectory, or streamline, of an ion/electron fluid element originating at a given radius from the center of the nozzle throat. The conservation of canonical angular momentum for each species, along with the continuity equation, yields the current along each streamline. Using interpolation between many streamlines, we obtain a current distribution in the meridional plane. Finally, finite differencing is used to determine the magnetic field induced by these currents. This process is then iterated upon until a magnetic field solution is found that is commensurate with the dynamics of the expanding plasma. More details about this model may be found in Ref. [7].

We previously found[7] that the expansion of the plasma according to our model depends mainly on four dimensionless parameters: the effective electron specific heat ratio, γ; the ratio of the plasma radius to the magnetic coil radius, $\hat{r}_{p,0}$, which we call nozzle divergence as it is a measure of the divergence of the magnetic field lines encountered by the flow; the ratio of the kinetic energy to the magnetic field energy, β_0; and the magnetization parameter, G given by

$$G = \frac{e^2 B^2 L^2}{m_e M_i U^2}. \tag{18}$$

Here, L and U are the characteristic length and velocity of the flow, respectively. This parameter is essentially the square of the ratio of the plasma radius to the Larmor radius of a hybrid particle of mass $m_H = (m_e M_i)^{1/2}$. As such, it may be seen as a measure of the relative magnetization of the flow.

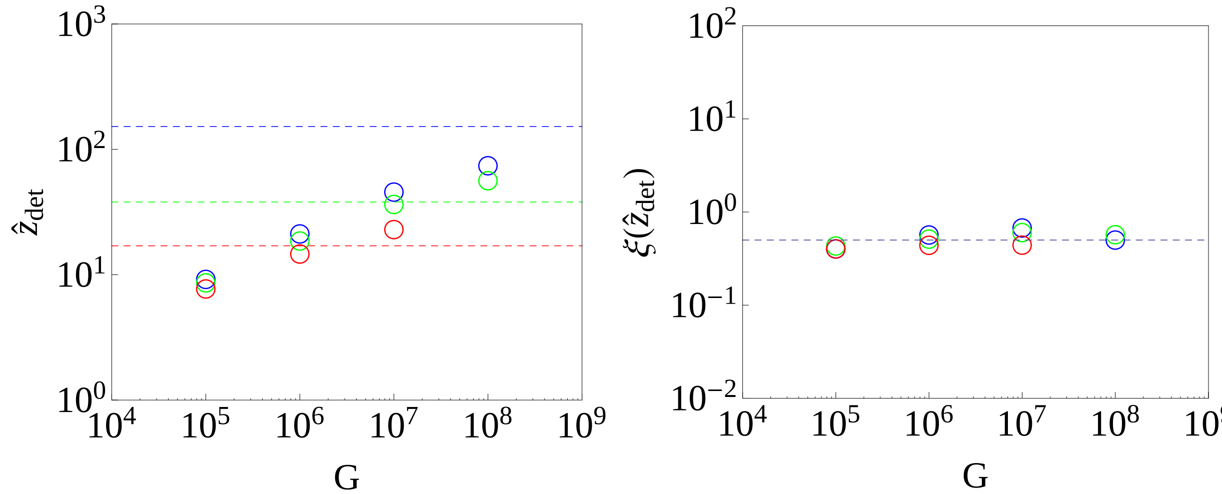

Figure 2. (a) Axial detachment location, $\hat{z}_{det}$, and (b) magnetization parameter at the detachment point, $\xi(\hat{z}_{det})$. Colors correspond to $\hat{r}_{p,0} = 0.05$ (blue), $\hat{r}_{p,0} = 0.10$ (green), and $\hat{r}_{p,0} = 0.15$ (red). Dashed lines in (a) correspond to the turning point of the initial flux surface of the plasma and (b) ξ=0.5.

Given these parameters, all of which are defined at the throat of the nozzle, the model solves for the acceleration and detachment of the flow downstream of this point. Simulations presented in this paper vary G and $\hat{r}_{p,0}$ for fixed values of γ and β_0.

Detachment due to the inertia of the flowing plasma was first investigated by Hooper.[3] In this scenario, the inertia of the downstream plasma increases relative to the effective inertia of the applied magnetic field. Eventually, the strength of the magnetic field decreases to the point where the ions and electrons separate from their initial magnetic flux surface. We found[7] this process to scale with G.

For the present study, we take the nozzle radius, r_c, as the characteristic length L, and the ion acoustic velocity at the throat, $c_{s,0}$, as the characteristic velocity U. The magnetic field strength is defined as its value at the center of the nozzle throat, B_0. In the results shown here, G varies by an order of magnitude between 10^5 and 10^8.

Three values of the normalized plasma radius, $\hat{r}_{p,0}$, are taken parametrically in this study: 0.05, 0.10, and 0.15. The electron specific heat ratio is assumed to be $\gamma = 1.2$. As we will see, the properties of the flow depend strongly on the chosen value of γ. As such, a detailed analysis of electron energy transport in an expanding plasma is required for a proper performance characterization but such an analysis is not necessary to answer the question we posed at the beginning of this section and will be left for future research. Finally, a value of $\beta_0 = 10^{-10}$ is taken so that induced magnetic fields may be ignored; however, the influence of induced fields on plasma detachment and momentum transfer will be commented upon.

IV. Simulation Results

In this section we use the results of our model to investigate the influence of inertial detachment on the generation of current within the expanding plasma for various magnetization parameters, G, and nozzle divergences, $\hat{r}_{p,0}$. First, the relative strength of the diamagnetic and paramagnetic currents are compared. The force per unit axial length that results from these currents is then used to quantify the mutual interaction between the plume and magnetic nozzle. Finally, the thrust coefficient and propulsion efficiency are presented along with an analysis of the nature of inertial detachment. Comments about the extrapolation of these results to other detachment scenarios are also included.

Detachment of the plasma from the applied magnetic field is observed for each case presented here. We define the axial detachment location, $\hat{z}_{det}$, as the location at which the plume divergence reaches 99% of its final value. This location, normalized by the nozzle radius, may be seen in Figure (5) for the various parameters studied in this paper. Also shown are dashed lines corresponding to the turning point of the initial flux surface of the plasma. As would be expected, detachment occurs further downstream as the plasma becomes more magnetized. Furthermore, $\hat{z}_{det}$ decreases as $\hat{r}_{p,0}$ decreases.

The magnetic nature of the plasma, i.e. whether it is diamagnetic or paramagnetic, strongly depends on the diamagnetic linear current density, J_θ, and the induced current density within the body of the plasma, j_θ. Force balance at the plasma-vacuum interface, Eq. (17), requires

$$J_\theta = -\frac{pu}{\mathbf{u}\cdot\mathbf{B}}, \tag{19}$$

where p is the pressure at the plasma edge. For a plasma expanding through an applied magnetic field, this current will always be diamagnetic such that it induces magnetic fields opposing the applied field. Thus, henceforth we will denote $J_{Diam} = J_\theta$.

The induced current density depends on the non-uniformity of the plasma density (diamagnetic) and the degree of separation of the plasma from its initial magnetic flux surface (paramagnetic). Our simulations consider the detachment of a uniform plasma, thus the current density will always be paramagnetic. Therefore, the plasma consists of a paramagnetic current density surrounded by a diamagnetic current layer.

We may then define a metric that characterizes the global magnetic nature of the plasma. The total dimensionless current per unit length at any axial location contained within the plasma is given by

$$\hat{J}_{Tot} \equiv \frac{\mu_0 J_{tot}}{B_0} = \hat{J}_{Diam} + \hat{J}_{Para} = -\frac{\beta_0}{\gamma}\frac{\hat{p}\hat{u}}{\hat{\mathbf{u}}\cdot\hat{\mathbf{B}}} + \int_0^{\hat{r}_p} \hat{j}_\theta d\hat{r}. \tag{20}$$

Lengths are normalized with respect to the nozzle coil radius, while other parameters are normalized by their value at the throat of the nozzle, i.e. $\hat{x} = x/x_0$. It may then be stated that, at a given axial location, the plasma is diamagnetic if $\hat{J}_{Diam} > \hat{J}_{Para}$ and paramagnetic if $\hat{J}_{Diam} < \hat{J}_{Para}$. We also note that Eq.(20) is an indication of the strength of the induced magnetic fields to the applied magnetic field, and that $\hat{j}_\theta$, and as a result $\hat{J}_{Tot}$, both scale linearly with β_0. In other words, induced magnetic fields become more important as $\hat{J}_{tot}$ approaches unity.

The dependence of the induced currents on the magnetization, G, may be seen in Figures 3(a) and 3(b), while the results for various nozzle divergences are shown in Figures 4(a) and 4(b). We have defined

$$\chi_{\hat{J}} = \left|\hat{J}_{Para}/\hat{J}_{Diam}\right| \tag{21}$$

to be the ratio of the diamagnetic current to the paramagnetic current. These figures show that, even though paramagnetic currents develop within the plume, a plasma of low magnetization and small nozzle divergence may detach while remaining globally diamagnetic throughout the plume ($\chi_{\hat{J}} < 1$). As the magnetization and divergence increase, however, the strength of the paramagnetic currents increases beyond the diamagnetic currents ($\chi_{\hat{J}} > 1$), and the plasma becomes globally paramagnetic. The following question is then raised: *is thrust adversely affected by the paramagnetic currents?*.

To answer this question we look at the normalized force per unit length, $\hat{f}_{Tot}$, transferred from the plasma to the nozzle. The thrust coefficient of a nozzle is conventionally defined as

$$C_F \equiv \frac{F}{p_{tot}A_t} = C_0 + \int_0^{\hat{z}} \hat{f}_{Tot} d\hat{z} = C_0 + \int_0^{\hat{z}} \hat{f}_{Diam} d\hat{z} + \int_0^{\hat{z}} \hat{f}_{Para} d\hat{z}. \tag{22}$$

Here, p_{tot} is the total pressure and A_t is the throat area. Furthermore, we have split up the thrust coefficient into its contributions due to the flow entering the throat, C_0, and due to the diamagnetic and paramagnetic forces per until length, $\hat{f}_{Diam}$ and $\hat{f}_{Para}$, respectively.

It is clear that $\hat{f}_{Para} < 0$ for a uniform plasma, thus we refer to this term as the paramagnetic drag on the nozzle coil. The requirement for positive thrust gain in the divergent portion of a magnetic nozzle then becomes

$$\left|\int_0^{\hat{z}} \hat{f}_{Diam} d\hat{z}\right| > \left|\int_0^{\hat{z}} \hat{f}_{Para} d\hat{z}\right|. \tag{23}$$

Explicit expressions for the diamagnetic and paramagnetic linear force densities can be found by normalizing $F_{L,V}^{(a)}$ and $F_{L,S}^{(a)}$ according to Eq.(22) and transforming the line integral along the plasma edge into an integral over $\hat{z}$. The following equations result:

$$\hat{f}_{Diam} = -\frac{2g(\gamma)}{\beta_0 \hat{r}_{p,0}^2}\hat{J}_{Diam}\hat{B}_r^{(a)}\hat{r_p}\sqrt{1+\left(\frac{\hat{u}_r}{\hat{u}_z}\right)^2}, \tag{24}$$

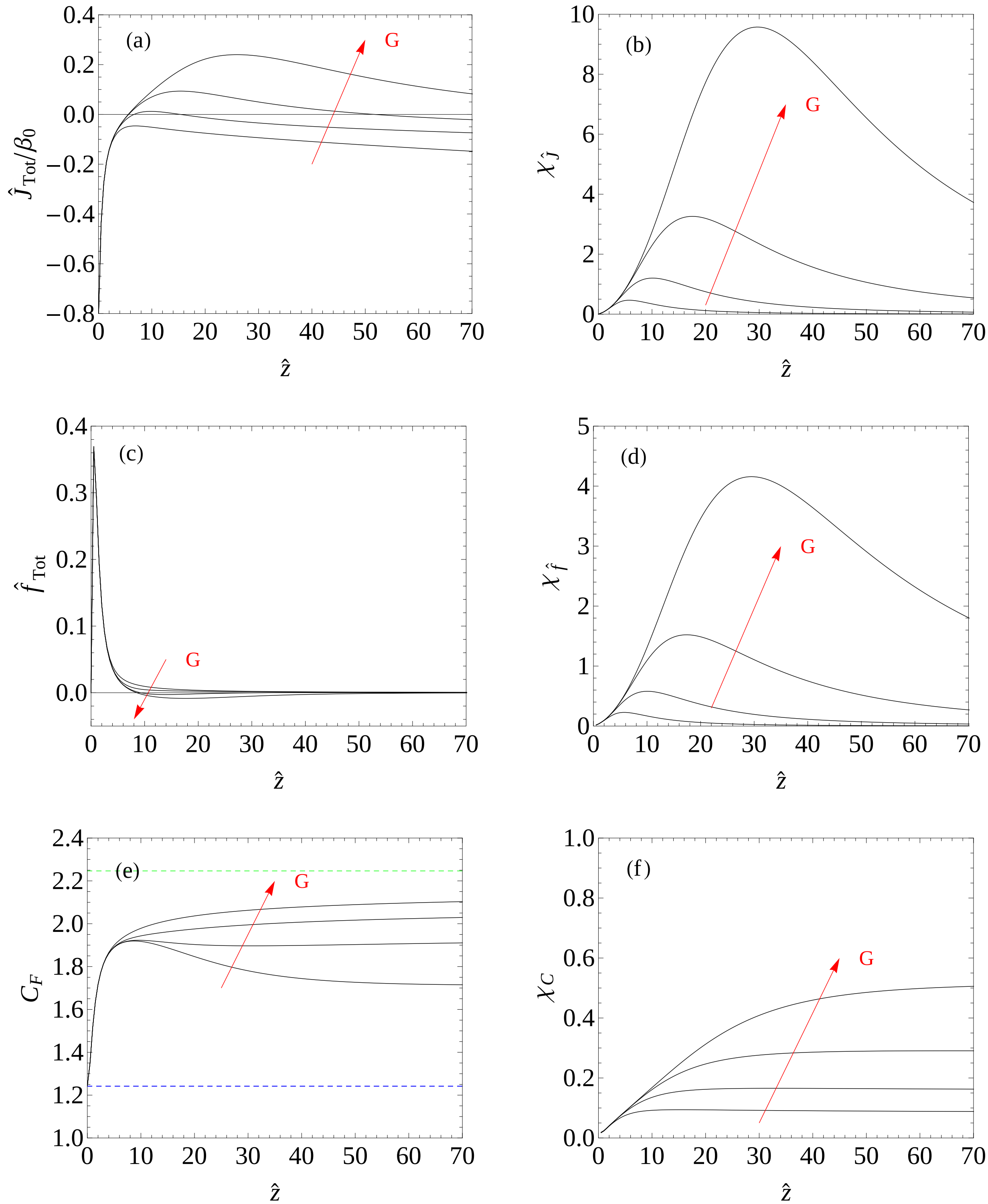

Figure 3. Axial evolution of: (a) Total dimensionless current divided by β_0, (b) ratio of paramagnetic current magnitude to diamagnetic current magnitude, (c) total linear force density, (d) ratio of paramagnetic linear force density magnitude to diamagnetic linear force density magnitude, (e) thrust coefficient with C_0 (blue) and $C_{F,max}$ (green), and (f) ratio of paramagnetic thrust coefficient magnitude to diamagnetic thrust coefficient magnitude. All figures are for $\hat{r}_{p,0} = 0.10$ and $G = \left(10^5, 10^6, 10^7, 10^8\right)$. Red lines indicate the direction of increasing G.

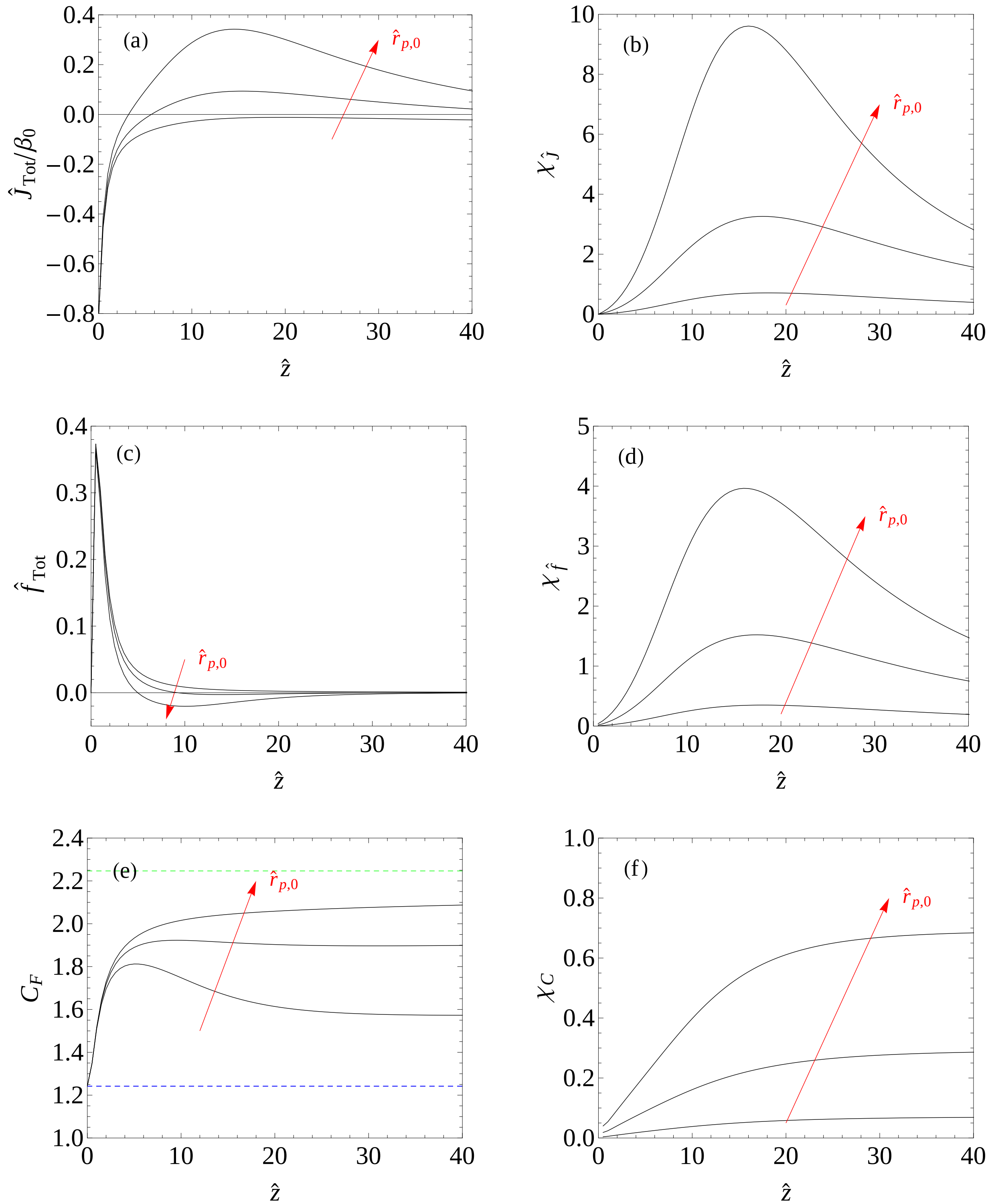

Figure 4. Axial evolution of: (a) Total dimensionless current divided by β_0, (b) ratio of paramagnetic current magnitude to diamagnetic current magnitude, (c) total linear force density, (d) ratio of paramagnetic linear force density magnitude to diamagnetic linear force density magnitude, (e) thrust coefficient with C_0 (blue) and $C_{F,max}$ (green), and (f) ratio of paramagnetic thrust coefficient magnitude to diamagnetic thrust coefficient magnitude. All figures are for $G = 10^7$ and $\hat{r}_{p,0} = (0.05, 0.10, 0.15)$. Red lines indicate the direction of increasing $\hat{r}_{p,0}$.

$$\hat{f}_{Para} = -\frac{2g(\gamma)}{\beta_0 \hat{r}_{p,0}^2} \int_0^{\hat{r}_p} \hat{j}_\theta \hat{B}_r^{(a)} \hat{r} d\hat{r}. \tag{25}$$

Furthermore, we have assumed that the expansion of the plasma through the convergent portion of the nozzle may be idealized as isentropic. This allows us to relate the total pressure to the pressure at the throat through the function,

$$g(\gamma) = \gamma \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}. \tag{26}$$

Eq.(23) is an extension of the requirement for positive thrust gain proposed by Ahedo and Moreno.[9] Thrust gain does not necessitate a globally diamagnetic plasma, rather, the integral of the force per unit length of the diamagnetic currents must dominate over the force per unit length of the paramagnetic currents. These force densities depend not only on the current, but also on the radial component of the applied magnetic field, $\hat{B}_r^{(a)}$, the radial extent of the plasma, $\hat{r}_p$, and the divergence of the plume, $\hat{u}_r/\hat{u}_z$.

The total force per unit length the plasma transfers to the nozzle coil and the ratio of its paramagnetic to diamagnetic components,

$$\chi_{\hat{f}} = \left|\hat{f}_{Para}/\hat{f}_{Diam}\right|, \tag{27}$$

for varying magnetization parameters may be seen in Figures 3(c) and 3(d), respectively. Similar plots for varying nozzle divergences are shown in Figures 4(c) and 4(d). The general trend in these figures suggest that the diamagnetic current layer transfers a majority of the force to the nozzle within a few nozzle radii from the throat. Following this point, the density decrease due to plasma expansion leads to a dramatic decrease in the strength of both the diamagnetic current layer and its force per unit length. Downstream from this region significant paramagnetic currents may develop as the plasma detaches. The paramagnetic force per unit length transferred to the coil by the detaching plasma is observed to be weaker than the diamagnetic force ($\chi_{\hat{f}} < 1$) throughout the entire plume for plasmas of low magnetization and divergence. On the other hand, increased magnetization and divergence lead to a significant drag attributed to the paramagnetic currents in the region of detachment ($\chi_{\hat{f}} > 1$).

The axial evolution of the thrust coefficient may be used to determine the consequence of paramagnetic drag on magnetic nozzle performance. The maximum thrust coefficient may be determined by setting the thrust power, $F^2/2\dot{m}$, equal to the flow power at the entrance to the throat. This yields,

$$C_{F,max} = g(\gamma) \sqrt{\left(\frac{\gamma+1}{\gamma-1}\right)}, \tag{28}$$

where we once again note the importance of the electron specific heat ratio. Furthermore, the thrust coefficient at the nozzle throat is given by

$$C_0 = \frac{\gamma+1}{\gamma} g(\gamma). \tag{29}$$

For propulsion applications we require that $C_F > C_0$. Furthermore, high efficiencies require the thrust coefficient be a significant fraction of its maximum possible value.

Figures 3(e) and 3(f) show how the thrust coefficient and the ratio of its paramagnetic to diamagnetic components,

$$\chi_C = \left|C_{Para}/C_{Diam}\right|, \tag{30}$$

evolve throughout the plasma plume for various magnetization parameters. Figures 4(e) and 4(f) provide similar plots for different nozzle divergences. A majority of the thrust gain occurs immediately downstream the nozzle throat, corresponding to the region of high diamagnetism. For some of the plasmas, a noticeable thrust loss is observed in the detachment region where the paramagnetic force outweighs the diamagnetic force; however, this loss is not enough to allow for $C_F < C_0$ in all of the plasmas in consideration. In general, it is observed that the thrust coefficient increases as the magnetization and fieldline divergence decrease.

The dependence of the asymptote of the thrust coefficient on the magnetization of the plasma and magnetic field divergence for a variety of entrance conditions may be seen in Figure 4(a). Also shown are the diamagnetic and paramagnetic components of the thrust coefficient. It is clear from this figure that variations in the thrust coefficient are due mainly to the paramagnetic component as opposed to the

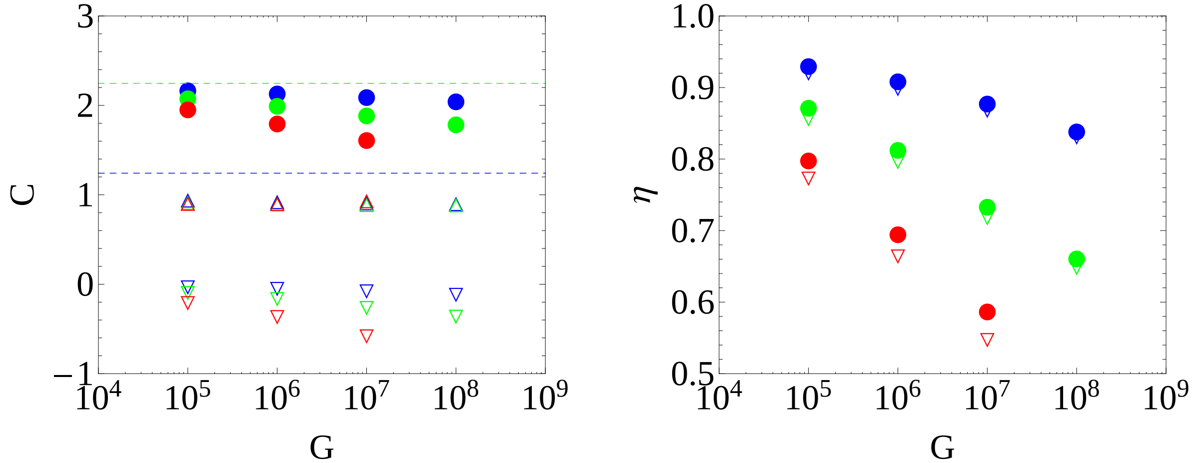

Figure 5. (a) Total thrust coefficient, C_F (solid circle), diamagnetic thrust coefficient, C_{Diam} (triangle), and paramagnetic thrust coefficient, C_{Para} (upside down triangle). (b) Total efficiency, η_{Tot} (solid circle), and divergence efficiency, η_{Div} (upside down triangle). Colors correspond to $\hat{r}_{p,0} = 0.05$ (blue), $\hat{r}_{p,0} = 0.10$ (green), and $\hat{r}_{p,0} = 0.15$ (red).

diamagnetic component, which is essentially independent of the incoming plasma. This trend becomes important when considering the propulsion and divergence efficiencies of the nozzle.

The propulsion efficiency of a thruster is conventionally defined as the thrust power divided by the input power to the thruster. For the purpose of this study, we neglect all efficiency losses up until the throat entrance. Consequently, the efficiency becomes $\eta_{Tot} = C_F^2/C_{F,max}^2$. The divergence efficiency, on the other hand, is defined at a given axial location as the ratio of the axially directed kinetic power to the total kinetic power of the flow. The propulsion and divergence efficiency trends in Figure 4(b) indicate that the prominent loss mechanism in the expansion and detachment of the magnetic nozzle plasma is plume divergence. As we have shown in Figure 3, the loss of thrust as viewed from the perspective of the nozzle is due mainly to the paramagnetic drag of the plasma. Therefore, a clear parallel exists between the divergence of the plasma and the relative strength of its paramagnetic drag. This result runs contrary to the claim that positive thrust gain ($C_F > C_0$) can only occur for detachment mechanisms that inhibit paramagnetic currents. Rather, paramagnetic currents may be generated due to detachment as long as the positive thrust gain requirement described in Eq.(23) is satisfied, i.e. as long as $\chi_C < 1$ as $\hat{z} \to \infty$.

We may generalize the conclusions presented above to both resistive and magnetic detachment. Similar to inertial detachment, the collisional diffusion of particles perpendicular to the magnetic field lines induces a current density within the plasma through the conservation of canonical angular momentum. While paramagnetic currents are induced if the plasma diffuses in a direction that ensures the plasma diverges at a rate less than the magnetic field, Figures 3 and 4 indicate that positive thrust gain is still possible and that increased diffusion, like decreased magnetization, would lead to improved efficiency.

Magnetic detachment, on the other hand, requires the consideration of induced magnetic fields. It may be stated, however, that the effect of induced magnetic fields is to limit the separation of the plasma from its initial flux surface. As such, the quantity $\hat{J}_{Para}/\beta_0$ would decrease along with the relative strength of the paramagnetic to diamagnetic currents, $\chi_{\hat{J}}$. As we saw for inertial detachment, this decrease generally implies lower plume divergences and improved thrust transmission and efficiency. Additional simulations accounting for the influence of induced magnetic fields on momentum transfer will be included in a future study.

An interesting phenomenon is seen when we relate the ratio of the hybrid Larmor radius of the plasma to the scale length of magnetic variation. We define this ratio as

$$\xi = G^{-1/2}\left|\frac{\hat{\nabla}\hat{B}}{\hat{B}}\right|, \tag{31}$$

and compute its value along the plasma edge at $\hat{z} = \hat{z}_{det}$. The result of this analysis may be seen in Figure 2(b). Evidently, the location of inertial detachment occurs at the point where $\xi \approx 0.50$. This is accurate to within about 40% for the results presented above. Therefore, inertial detachment may be viewed as the gradual demagnetization and seperation of a hybrid particle from its initial flux surface.

Recently, Ahedo and Merino[10] showed that the local ambipolarity assumption leading to the hybrid particle formulation breaks down for certain flows. In their analysis, the electrons are the dominant species for detachment. For this scenario, the detachment parameter in Eq. (31) would have to be replaced by an effective value, $G_{eff} = (M_i/m_e)G$. Thus, detachment would be significantly hindered. Their results, however, ignore electron inertia and require that the electrons remain tied to their initial flux surface throughout the plume. This is very different from our assumption, which includes electron inertia and electron detachment, but requires the electron motion be tied to the ion motion. The domain of validity for each assumption is beyond the scope of this paper and will be the topic of a future study.

V. Conclusions

Using Green's functions, we have proved that the transfer of momentum from the plasma to the thruster occurs through the mutual interaction between plasma currents and the current within the applied magnetic field coil. The force generated from this reaction is equal and opposite to the volumetric sum of the Lorentz force density due to the *applied* magnetic field acting on the plasma currents. A two-fluid model for the expansion of a fully ionized plasma was used to study the influence of plasma currents induced through inertial detachment on the transmission of thrust. Contrary to previous belief, positive thrust gain was observed for plasma plumes that were paramagnetic in nature. Furthermore, any decrease in performance due to the mutual attraction between the plasma and nozzle was observed to be synonymous with the decrease in performance due to plume divergence losses. Finally, we found that inertial detachment may be viewed as the gradual demagnetization of a hybrid particle of mass, $m_H = (m_e M_i)^{1/2}$, whereby its Larmor radius becomes the same order as the scale length of magnetic field variation.

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